The Character Table of the Nonabelian Group of Order pq

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Abstract

In this paper, I discuss the representation theory, and fill out the character table, of the unique nonabelian group of order pq for primes $p < q$ with $q \equiv 1 \pmod{p}$. The first section of the paper recalls the basic properties of semidirect products, and then classifies all groups of order pq for primes $p < q$, proving that there are exactly two isomorphism classes–one $\mathbb{Z}/pq\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ and, only when $q \equiv 1 \pmod{p}$, one nonabelian semidirect product group $G = \mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$. Then, I give two concrete ways to realize G: as a subgroup of the symmetric group $\mathfrak{S}_q$ and as an index-$(q-1)/p$ subgroup of $\mathrm{Aff}(\mathbb{A}^1\mathbb{F}_q)$, the group of affine transformations of the affine line over the field with q elements of order $q(q-1)$. Next, I discuss the conjugacy structure of G, and finally, I fill out the character table of G, using tools from Chapters 1 to 3 of [1], specifically induced representations and Frobenius reciprocity. This was an enlightening computation for me, because this was what really made the representation theory of finite groups "click" for me.

Contents

1 Introduction

We begin by recalling basic properties of the semidirect product, as in [2] §5.5. Let H and K be groups and $\Psi : K \to \mathrm{Aut}(H)$, $k \mapsto \Psi_k$ be a group homomorphism. Then the *semidirect product group* $H \rtimes_\Psi K$ is defined to be the set $\{(h,k) : h \in H, k \in K\}$ with law of composition

$$(h,k)\cdot(h',k') = (h\Psi_k(h'), kk').$$

This law of composition makes $H \rtimes_\Psi K$ a group with identity $e = (e_H, e_K)$ and inverses given by $(h,k)^{-1} = (\Psi_{k^{-1}}(h^{-1}), k^{-1})$. The maps $H \to H \rtimes_\Psi K : h \mapsto (h, e_K)$ and $K \to H \rtimes_\Psi K : k \mapsto (e_H, k)$ are injective homomorphisms, allowing us to identify H and K as subgroups $H, K \leq H \rtimes_\Psi K$. Further, under this identification, we have that

(1) $H \trianglelefteq H \rtimes_\Psi K$,
(2) $H \cap K = \{e\}$,
(3) $|H \rtimes_\Psi K| = |H| \cdot |K|$,
(4) for any $h \in H$ and $k \in K$ we have that $khk^{-1} = \Psi_k(h) \in H \rtimes_\Psi K$,
(5) there is a split short exact sequence $\{e\} \to H \to H \rtimes_\Psi K \to K \to \{e\}$, and conversely every split short exact sequence $\{e\} \to H \to G \to K \to \{e\}$ identifies $G \cong H \rtimes_\Psi K$ for some $\Psi : K \to \mathrm{Aut}(H)$.

This is actually a generalization of the direct product:

Lemma 1. *For a semidirect product $H \rtimes_\Psi K$, the following are equivalent:*

(1) The identity set map $H \times K \to H \rtimes_\Psi K$ is a group homomorphism (and hence an isomorphism).
(2) The map $\Psi : K \to \mathrm{Aut}(H)$ is trivial.
(3) The subgroup $K \trianglelefteq H \rtimes_\Psi K$ is normal.

If different nontrivial Ψ give the same group up to isomorphism, then it is denoted by $H \rtimes K$. We identify semidirect products in practice using the Recognition Theorem.

Theorem 1 (Recognition Theorem)**.** *Suppose G is a finite group with subgroups $H, K \leq G$ s.t.*

(1) $H \trianglelefteq G$,
(2) $H \cap K = \{e\}$, and
(3) $|G| = |H| \cdot |K|$.

Then G is the semidirect product of H and K. If $\Psi : K \to \mathrm{Aut}(H)$ is given by $k \mapsto (h \mapsto khk^{-1})$, then we have $G \cong H \rtimes_\Psi K$.

Proof Sketch. Consider the set map $H \rtimes_\Psi K \to G : (h,k) \mapsto hk$. It is immediate to see that this is a homomorphism. Condition (2) tells us that this map is injective, and then condition (3) tells us that it is an isomorphism. ■

Using the above recognition theorem and some Sylow theory, it is possible to classify all groups of order pq for primes $p < q$.

Theorem 2 (Classification of Groups of Order pq)**.** *Suppose G is a finite group of order $|G| = pq$ for distinct primes $p < q$. Then*

(1) if $p \nmid q-1$, then $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$ is cyclic,
(2) if $p \mid q-1$, then either $G \cong \mathbb{Z}/pq\mathbb{Z}$ is cyclic, or it is the unique nontrivial semidirect product $\mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.

Proof. Let G be a group of order pq for primes $p < q$. Let s_p and s_q denote the number of Sylow p-subgroups and Sylow q-subgroups of G respectively. Sylow's Third Theorem tells us that

$$s_q \equiv 1 \pmod{q} \text{ and } s_q \mid p.$$

This is only possible if $s_q = 1$. This tells us that there is a unique (and hence normal) subgroup $H \trianglelefteq G$ of order q. Again, Sylow's Third Theorem tells us that

$$s_p \equiv 1 \pmod{p} \text{ and } s_p \mid q.$$

Therefore, either $s_p = 1$ or $s_p = q$.

(1) If $s_p = 1$, then G also has a unique normal subgroup $K \trianglelefteq G$ of order p. Since $(p,q) = 1$, we must have $H \cap K = \{e\}$. Then H and K satisfy hypotheses of Theorem 1 and Lemma 1, so that $G \cong H \times K \cong \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(2) If $s_p = q$, then we must have $p \mid q-1$. Let K be any Sylow p-subgroup. Then Theorem 1 tells us that $G \cong H \rtimes_\Psi K$ for some homomorphism $\Psi : K \to \operatorname{Aut}(H)$. To analyze such Ψ, pick generators $H = \langle a \rangle$ with $|a| = q$ and $K = \langle b \rangle$ with $|b| = p$. Now the map $H \rtimes_\Psi K \to G : (h,k) \mapsto hk$ is an isomorphism, so every element of G can be written as $a^i b^j$ for some $0 \le i \le q-1$ and $0 \le j \le p-1$. The conjugate $bab^{-1} \in H$, so that $bab^{-1} = a^\lambda$ for some $\lambda \in \mathbb{F}_q^\times$. If $\lambda = 1$, then a and b commute, so that we have $G \cong H \times K$, which would imply $s_p = 1$, which is not the case we are in. Therefore, $\lambda \in \mathbb{F}_q^\times \smallsetminus \{1\}$. Then we must have

$$a = b^p a b^{-p} = a^{\lambda^p} \Rightarrow \lambda^p = 1 \in \mathbb{F}_q^\times,$$

which means that $\lambda \in \mathbb{F}_q^\times$ has order exactly p. This completely determines the Cayley table, so that we get that G has presentation

$$G = \left\langle a, b \,\middle|\, a^q = b^p = e,\, bab^{-1} = a^\lambda \right\rangle.$$

To see that a different choice of λ would not change the isomorphism type: pick any other element say $\mu \in \mathbb{F}_q^\times$ of order p. Since the subgroup of elements of $\mathbb{F}_q^\times$ of order dividing p is cyclic, we get that $\mu = \lambda^k$ for some $1 \le k \le p-1$. Then picking the generator $c = b^k$ of K instead of b, we would get the presentation with $cac^{-1} = a^\mu$; and this tells us that the all different nontrivial Ψ give the same semidirect product.

■

The nontrivial semidirect product is given by $\Psi : \mathbb{Z}/p\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$ sending $\overline{1}$ to any element of order p in $\mathbb{F}_q^\times$. This gives the group presentation

$$G = \left\langle a, b \,\middle|\, a^q = b^p = e,\, bab^{-1} = a^\lambda \right\rangle,$$

where $\lambda \in \mathbb{F}_q^\times$ has order p. One concrete way to get such a λ is to pick a primitive root r mod q and then take $\lambda = r^{\frac{q-1}{p}}$.

Remark 1. For convenience, I henceforth use the letter t to denote $(q-1)/p$, so that $q = pt+1$, and we can choose $\lambda = r^t$.

2 Two Realizations of G

(1) Firstly, G can be realized as a subgroup $G \le \mathfrak{S}_q$ as follows. Since a has order q, we try $a \mapsto \sigma := (1,2,3,\cdots,q)$. Then $bab^{-1} = a^\lambda$ forces us to send $b \mapsto \tau := (j \mapsto 1 + (j-1)\lambda \pmod q)$; note that τ is indeed a permutation with inverse $\tau^{-1} = (j \mapsto 1 + (j-1)\lambda^{-1} \pmod q)$. If we choose a primitive root r mod q so that $\lambda = r^t$, then τ keeps 1 fixed, and decomposes into a product of t cycles of length p as follows

$$\tau = \prod_{j=0}^{t-1} (r^j + 1, r^{t+j} + 1, \cdots, r^{(p-1)t+j} + 1).$$

This realizes G as a subgroup

$$G \cong \langle \sigma, \tau \rangle \le \mathfrak{S}_q.$$

(2) Another way to realize G is as an index-t subgroup of $\operatorname{Aff}(\mathbb{A}^1\mathbb{F}_q)$. Recall that for any field $\mathbb{F}$, the group

$$\operatorname{Aff}(\mathbb{A}^1\mathbb{F}) = \{x \mapsto \alpha x + \beta : \mathbb{A}^1\mathbb{F} \to \mathbb{A}^1\mathbb{F} \,|\, \alpha, \beta \in \mathbb{F}, \alpha \neq 0\}$$

is the group of affine transformations of the affine line over the field $\mathbb{F}$. It can also be realized as a subgroup $\operatorname{Aff}(\mathbb{A}^1\mathbb{F}) \le \operatorname{GL}_2\mathbb{F}$ via $(x \mapsto \alpha x + \beta) \mapsto \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}$. If $\mathbb{F}$ is a finite field $\mathbb{F} = \mathbb{F}_q$, then the order $|\operatorname{Aff}(\mathbb{A}^1\mathbb{F}_q)| = (q-1)q$. Sitting inside it is the subgroup of affine transformations $x \mapsto \alpha x + \beta$

with $\alpha \in \mathbb{F}_q$ of order p in $\mathbb{F}_q^\times$, i.e. satisfying $\alpha^p = 1 \in \mathbb{F}_q$. Again, it is immediate to see that the map $G \to \mathrm{Aff}(\mathbb{A}^1\mathbb{F}_q)$ given by $a \mapsto (x \mapsto x+1)$ and $b \mapsto (x \mapsto \lambda x)$ is an injective group homomorphism, realizing

$$G \leq \mathrm{Aff}(\mathbb{A}^1\mathbb{F}_q)$$

as a normal subgroup $G \trianglelefteq \mathrm{Aff}(\mathbb{A}^1\mathbb{F}_q)$ with quotient cyclic of order t, generated by the class of $x \mapsto rx$ for any primitive root r mod q.

3 Conjugacy Structure

The next step is to find the conjugacy structure of the group $G = \langle a, b | a^q = b^p = e, bab^{-1} = a^\lambda \rangle$, where $\lambda \in \mathbb{F}_q^\times$ has order p. Recall that every element of G can be written uniquely as $a^i b^j$ for $0 \leq i \leq q-1$ and $0 \leq j \leq p-1$.

(1) Let's figure out the center $Z(G)$. Suppose $a^i b^j \in Z(G)$ for some $0 \leq i \leq q-1$ and $0 \leq j \leq p-1$. Then

$$a^i b^j = a^{i+1} b^j a^{-1} = a^{i+1-\lambda^j} b^j,$$

so that $i \equiv i + 1 - \lambda^j \pmod q \Rightarrow \lambda^j \equiv 1 \pmod q \Rightarrow j = 0$ since λ has order p in $\mathbb{F}_q^\times$. Then

$$a^i = ba^i b^{-1} = a^{\lambda i},$$

so that $i \equiv \lambda i \pmod q$ and hence $i = 0$. This means that $Z(G) = \{e\}$, i.e. the center of G consists of the identity alone.

(2) The class equation of G looks like

$$pq = 1 + mp + nq$$

for some integers m and n with $0 \leq m \leq q-1$ and $0 \leq n \leq p-1$. Reducing mod p, we get that $n \equiv -1 \pmod p$, so that we must have $n = p-1$. Then $m = t$. Therefore G has exactly $1 + m + n = p + t$ distinct conjugacy classes, and hence irreducible representations.

(3) Let's find out the conjugacy classes. For that, pick a primitive root r mod q so that $\lambda = r^t$. The relation $bab^{-1} = a^{r^t}$ tells us that conjugating by b (and also of course by a) doesn't change the congruence class of j modulo t for an element in G of the form a^{r^j}, so that the t conjugacy classes of order p are those of a^{r^j} for $0 \leq j \leq t-1$. The conjugacy class of a^{r^j} comprises of $a^{r^j}, a^{r^{t+j}}, \cdots, a^{r^{(p-1)t+j}}$. Again, the relations

$$a(a^i b^j)a^{-1} = a^{i+1-\lambda^j} b^j \text{ and } b(a^i b^j)b^{-1} = a^{\lambda i} b^j,$$

tell us that conjugating by any element of G doesn't change the exponent of b in an element of G, so that the $p-1$ classes of order q are those of b^i for $1 \leq i \leq p-1$. Therefore, a complete set of representatives of conjugacy classes is given by

$$e, b, b^2, \cdots, b^{p-1}, a, a^r, a^{r^2}, \cdots, a^{r^{t-2}} \text{ and } a^{r^{t-1}}.$$

4 Character Table

Finally, let's fill in the character table of G. We expect there to be exactly $p+t$ distinct irreducible representations of G. We know the first one: the trivial representation. In fact, we know much better: the normal subgroup $H = \langle a \rangle \leq G$ has quotient $G/H \cong \langle b \rangle \cong \mathbb{Z}/p\mathbb{Z}$, which has p distinct 1-dimensional irreducible representations U_k for $0 \leq k \leq p-1$, where b acts on U_i as ζ_p^k for $\zeta_p = \exp(2\pi i/p)$ a primitive p^{th} root of unity. Pulling these back to G gives p distinct 1-dimensional irreducible representations U_k of G; further, these are all the 1-dimensional representations of G, since a given 1-dimensional representation must factor through the abelianization of G and hence be trivial on the classes of powers of a. Therefore, the character table looks so far like:

G $\langle a, b \vert a^q, b^p, bab^{-1}a^{-r^t} \rangle$	1 e	q b^i $1 \leq i \leq p-1$	p a^{r^j} $0 \leq j \leq t-1$
U_k $0 \leq k \leq p-1$	1	ζ_p^{ki}	1

Observe that in these terms, the representation U_0 is the trivial representation of G.

We're now missing t representations, say $V_0, \cdots, V_{t-1}$ of dimensions $d_0, \ldots, d_{t-1} \geq 2$, respectively. Then [1] Equation 2.19 p 17 gives

$$pq = \sum_{k=0}^{p-1} 1^2 + \sum_{\ell=0}^{t-1} d_\ell^2 = p + \sum_{\ell=0}^{t-1} d_\ell^2,$$

where each $d_\ell \in \{p, q\}$ by [1] Problem 2.38 p. 25. Therefore, this equation looks like

$$pq = p + mp^2 + nq^2$$

for some integers $0 \leq m, n$ with $m + n = t$. Since $q > p$, we must have $n = 0$ and $m = t$. Therefore, $d_0 = d_1 = \cdots = d_{t-1} = p$, i.e. all the remaining representations have dimension p. How do we nail these t representations down?

Let's first see if we can squeeze anything more out of the orthonormality of characters. For any $k : 0 \leq k \leq p-1$ and $\ell : 0 \leq \ell \leq t-1$, we have that

$$0 = \langle \chi_{U_k}, \chi_{V_\ell} \rangle = \frac{1}{pq} \left(p + \sum_{i=1}^{p-1} \zeta_p^{-ik} \chi_{V_\ell}(b^i) + \sum_{j=0}^{t-1} \chi_{V_\ell}(a^{r^j}) \right)$$

Summing over all k, we get that

$$\begin{aligned} 0 &= \sum_{k=0}^{p-1} \left(p + \sum_{i=1}^{p-1} \zeta_p^{-ik} \chi_{V_\ell}(b^i) + \sum_{j=0}^{t-1} \chi_{V_\ell}(a^{r^j}) \right) \\ &= p^2 + \sum_{i=0}^{p-1} \chi_{V_\ell}(b^i) \sum_{k=0}^{p-1} \zeta_p^{-ik} + p \sum_{j=0}^{t-1} \chi_{V_\ell}(a^{r^j}) \\ &= p \left(p + \sum_{j=0}^{t-1} \chi_{V_\ell}(a^{r^j}) \right), \end{aligned}$$

since the sum of all k^{th} roots of unity in any order is zero. This means that for each $k : 0 \leq k \leq p-1$,

$$\sum_{i=1}^{p-1} \zeta_p^{-ik} \chi_{V_\ell}(b^i) = 0.$$

Fix a particular $j : 1 \leq j \leq p-1$. Multiplying the above equality by ζ_p^{jk} and summing over all k gives

$$0 = \sum_{k=0}^{p-1} \zeta_p^{jk} \sum_{i=1}^{p-1} \zeta_p^{-ik} \chi_{V_\ell}(b^i) = \sum_{i=1}^{p-1} \chi_{V_\ell}(b^i) \sum_{k=0}^{p-1} \zeta_p^{(j-i)k} = \sum_{i=1}^{p-1} \chi_{V_\ell}(b^i) \cdot p\delta_{ij} = p\chi_{V_\ell}(b^j),$$

so that in fact for each $\ell : 0 \leq \ell \leq t-1$ and $i : 0 \leq i \leq p-1$ we get that $\chi_{V_\ell}(b^i) = 0$. This is expected, because the character of any irreducible representation of dimension at least 2 takes the value 0 on some conjugacy class, from [1] Problem 2.39 p. 25. Therefore, the character table looks like:

G $\langle a, b \mid a^q, b^p, bab^{-1}a^{-r^t} \rangle$	1 e	q b^i $1 \leq i \leq p-1$	p a^{r^j} $0 \leq j \leq t-1$
U_k $0 \leq k \leq p-1$	1	ζ_p^{ki}	1
V_ℓ $0 \leq \ell \leq t-1$	p	0	?

What replaces the question mark above?

5 Induction and Conclusion

The key to answering this question is induction of representations and Frobenius Reciprocity. Recall that we have a normal subgroup $H = \langle a \rangle \leq G$ of order q. This has exactly q distinct 1-dimensional irreducible representations W_s for $0 \leq s \leq q-1$, where a acts on W_s as ζ_q^s with $\zeta_q = \exp(2\pi i/q)$ a primitive q^{th} root of unity. For each $\ell : 0 \leq \ell \leq t-1$, the restriction $\operatorname{Res}_H^G V_\ell$ is p-dimensional, and therefore decomposes as

$$\operatorname{Res}_H^G V_\ell = W_{s_{\ell,1}} \oplus W_{s_{\ell,2}} \oplus \cdots \oplus W_{s_{\ell,p}}$$

for integers $0 \leq s_{\ell,1} \leq s_{\ell,2} \leq \cdots \leq s_{\ell,p} \leq q-1$. Since the character of V_ℓ is not identically p, we must have that $1 \leq s_{\ell,p}$. Since $W_{s_{\ell,p}}$ occurs in $\operatorname{Res}_H^G V_\ell$, Frobenius Reciprocity tells us that V_ℓ occurs in $\operatorname{Ind}_H^G W_{s_{\ell,p}}$. But now

$$\dim V_\ell = p = |G/H| \dim W_{s_{\ell,p}} = \dim \operatorname{Ind}_H^G W_{s_{\ell,p}},$$

so that in fact we must have $V_\ell = \operatorname{Ind}_H^G W_{s_{\ell,p}}$. Hence, it suffices to analyze $\operatorname{Ind}_H^G W_s$ for $1 \leq s \leq q-1$.

For any $s : 1 \leq s \leq q-1$, we compute the character $\chi_{\operatorname{Ind}_H^G W_s}$. We know that it must take value 0 on the class of any power of b, so it suffices to analyze $\chi_{\operatorname{Ind}_H^G W_s}(a^{r^j})$, for which we use [1] Equation (3.18) p. 34 to get

$$\chi_{\operatorname{Ind}_H^G W_s}(a^{r^j}) = \sum_{\substack{\sigma \in G/H \\ a^{r^j}\sigma = \sigma}} \chi_{W_s}(g_\sigma^{-1} a^{r^j} g_\sigma),$$

where the sum is over cosets σ that are preserved under left multiplication by a^{r^j}, and $g_\sigma \in G$ is any representative of σ. Since *every* coset in G/H is preserved by left multiplication by any power of a, we in fact get that

$$\chi_{\operatorname{Ind}_H^G W_s}(a^{r^j}) = \sum_{i=0}^{p-1} \chi_{W_s}(b^i a^{r^j} b^{-i}) = \sum_{i=0}^{p-1} \chi_{W_s}(a^{r^{it+j}}) = \sum_{i=0}^{p-1} \zeta_q^{sr^{it+j}}.$$

Since r is a primitive root, we must have $s = r^u$ for some $u : 0 \leq u \leq q-1$. Then, we see that

$$\chi_{\operatorname{Ind}_H^G W_s}(a^{r^j}) = \sum_{i=0}^{p-1} \zeta_q^{r^{it+u+j}},$$

which depends only on the congruence class $u \pmod t$. Therefore, this gives us exactly t distinct values, as we expected, for the characters of $V_\ell : 0 \leq \ell \leq t-1$. WLOG, we may relabel these so that

$$\chi_{V_\ell}(a^{r^j}) = \sum_{i=0}^{p-1} \zeta_q^{r^{it+\ell+j}},$$

and with this, we complete the character table:

G $\langle a, b \mid a^q, b^p, bab^{-1}a^{-r^t}\rangle$	1 e	q b^i $1 \leq i \leq p-1$	p a^{r^j} $0 \leq j \leq t-1$
U_k $0 \leq k \leq p-1$	1	ζ_p^{ki}	1
V_ℓ $0 \leq \ell \leq t-1$	p	0	$\sum_{i=0}^{p-1} \zeta_q^{r^{it+\ell+j}}$

Remark 2. In the above calculation, we encounter Gauss's "periods" of roots of unity: the different $\sum_{i=0}^{p-1} \zeta_q^{r^{it+\ell}}$ for $0 \leq \ell \leq t-1$. In the special case when $p = 2$ and q is a Fermat prime $q = 2^{2^n} + 1$ for some $n \geq 0$, Gauss was able to use these "two-member periods" $\zeta_q^u + \zeta_q^{-u}$ for $0 \leq u \leq 2^{2^n-1} - 1$ to prove his famous constructability result for regular polygons, as explained in [3] Chapter 7. These periods have fascinating properties, especially related to the action of $\operatorname{Gal}(\mathbb{Q}[\zeta_q]/\mathbb{Q})$. For instance, the product of two such periods can be written as a sum of these, as explained in [3] §7.2. With our tools, we can recognize this statement following from the complete reducibility of the tensor product of two of the V_ℓ's. This can be used to prove, for instance, that the unique quadratic subfield of $\mathbb{Q}[\zeta_q]$ is $\mathbb{Q}[\sqrt{q^*}]$, where $q^* = (-1)^{(q-1)/2}q$.

Remark 3. The above reasoning explains what the induced representations $\operatorname{Ind}_H^G W_s$ for $1 \leq s \leq q-1$ look like. What about $s = 0$? It is immediate to check using Frobenius Reciprocity that in fact,

$$\operatorname{Ind}_H^G W_0 = \bigoplus_{k=0}^{p-1} U_k.$$

This completes our understanding of induction of representations from H to G.

Remark 4. In the above process, we noticed that for this case of $G = H \rtimes K$, the irreducible representations were of two kinds: those pulled back from the quotient $G \twoheadrightarrow K$ and those induced from the subgroup $H \hookrightarrow G$. Serre's "Linear Representations of Finite Groups" [4] §8.2 gives a vast generalization of the above method: it describes the method of "little groups" by Wigner and Mackey that can be used to classify all irreducible representations of groups of the form $G = H \rtimes_\Psi K$ for $H \trianglelefteq G$ abelian, given that we know the representations of K.

References

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