An Introduction to Clifford Algebras and Spin Groups

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Abstract

This paper is written in fulfilment of the requirements of Math 262A: Quantum Theory from a Geometric Perspective taught at Harvard in the Fall 2023 semester by Prof. Dan Freed. In this paper, we develop the basic theory of Clifford algebras and the (s)pin groups, and build up to the result that the natural map $\rho : \mathrm{Spin}_n \to \mathrm{SO}_n$ gives us an explicit realization of the universal cover of SO_n for $n \geq 3$.

Contents

1 A Little Superalgebra

Throughout, we work over a fixed field k of characteristic other than 2. We assume that all algebras are unital and associative.

1.1 Super Vector Spaces

Lemma/Definition 1.1.1. On a vector space V over k, the following pieces of data are equivalent:

(a) a $\mathbb{Z}/2$-grading: subspaces $V^0, V^1 \subset V$ such that V is the internal direct sum of V^0 and V^1, i.e. such that the natural map $V^0 \oplus V^1 \to V$ is an isomorphism;
(b) an endomorphism $\alpha_V : V \to V$ such that $\alpha_V^2 = 1_V$; and
(c) a lift of the k-module structure on V to a $k[x]/(x^2-1)$-module structure.

A k-vector space V with this additional structure is called a super vector space over k. Further:

- The endomorphism α_V of (b) above is called the grading morphism of the super vector space V.
- Elements of $V^0 \cup V^1$ are called homogenous and for a nonzero homogenous $v \in V$, we define the parity of v to be $i \in \mathbb{Z}/2$ if $v \in V^i$, and denote it by $|v|$.

Proof. For (a) $\Leftrightarrow$ (b), from a $\mathbb{Z}/2$-grading $V^0, V^1 \subset V$, define $\alpha_V := 1_{V^0} \oplus -1_{V^1}$, and from an α_V, recover V^i for $i \in \mathbb{Z}/2$ as $(-1)^i$-eigenspaces of α_V.[1] The structures in (b) and (c) are clearly equivalent. ∎

Definition 1.1.2. Let V, W be super vector spaces over k. A morphism of super vector spaces $T : V \to W$ is a k-linear map that satisfies the following equivalent conditions:

(a) T respects the $\mathbb{Z}/2$-grading, i.e. $T(V^i) \subset W^i$ for $i \in \mathbb{Z}/2$;
(b) T commutes with the grading morphisms, i.e. that $T\alpha_V = \alpha_W T$; and
(c) T lifts to a $k[x]/(x^2-1)$-module homomorphism.

We denote by SVec_k be the category of super vector spaces over k.

Remark 1. When $\operatorname{char} k = 2$, the accepted definition is the one in (a).

Definition 1.1.3. If V and W are super vector spaces over k, then their tensor product $V \otimes W$ acquires a super vector space structure via

$$(V \otimes W)^k = \bigoplus_{i+j=k} V^i \otimes W^k$$

for $i, j, k \in \mathbb{Z}/2$. This amounts to taking $\alpha_{V\otimes W} = \alpha_V \otimes \alpha_W$. If we define the swap map

$$s_{VW} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$$

for homogenous $v \in V$ and $w \in W$, then this gives SVec_k the structure of a symmetric monoidal category with the unit being k, thought of as a super vector space with $k^0 = k$ and $k^1 = 0$.

[1] The diagonalizability of an endomorphism α_V satisfying $\alpha_V^2 = 1$ needs $\operatorname{char} k \neq 2$. A simple counterexample where this fails in $\operatorname{char} k = 2$ is provided by taking $V = k^2$ and $\alpha_V := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

1.2 Superalgebras

Definition 1.2.1.

(a) A superalgebra A over k, or a k-superalgebra, is a super vector space A over k that is a k-algebra such that the k-linear multiplication map $A \otimes A \to A$ is a morphism of super vector spaces, i.e. such that

$$A^i A^j \subset A^{i+j}$$

for $i, j \in \mathbb{Z}/2$.

(b) A morphism of k-superalgebras $f : A \to B$ is a k-algebra homomorphism that is also a linear map of super vector spaces over k.

We let SAlg_k denote the category of k-superalgebras.

(c) Let A be a k-superalgebra. The supercommutator of two elements $x, y \in A$ is defined for homogenous x and y as

$$[x, y] := xy - (-1)^{|x||y|} yx$$

and extended to be bilinear. The center of A is defined to be

$$Z(A) := \{x \in A : [x, y] = 0 \text{ for all } y \in A\}.$$

A superalgebra A is said to be supercommutative if $Z(A) = A$.

(d) If A and B are two k-superalgebras, then the tensor product $A \otimes B$ obtains the structure of a superalgebra determined by the rule

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|}(a_1 a_2 \otimes b_1 b_2)$$

for homogenous $a_1, a_2 \in A$ and $b_1, b_2 \in B$. The resulting k-superalgebra is called the super tensor product of A and B and is denoted $A \hat{\otimes} B$.

Here's a universal property satisfied by the supertensor product, the proof of which is clear:

Lemma 1.2.2. Let A and B be k-superalgebras. Then there are natural k-superalgebra morphisms $\iota_A : A \to A\hat{\otimes}B$ and $\iota_B : B \to A\hat{\otimes}B$ given by $\iota_A(a) = a \otimes 1$ and $\iota_B(b) = 1 \otimes b$. These have the property that if $a \in A$ and $b \in B$ are any elements, then their supercommutator in $A\hat{\otimes}B$ vanishes, i.e.

$$[\iota_A(a), \iota_B(b)] = 0.$$

Further, $A\hat{\otimes}B$ is universal with respect to this property, i.e. if C is any k-superalegbra and $\varphi_A : A \to C$ and $\varphi_B : B \to C$ morphisms of k-superalgebras such that for all $a \in A$ and $b \in B$ we have that

$$[\varphi_A(a), \varphi_B(b)] = 0,$$

then there is a unique k-superalgebra morphism $\varphi : A\hat{\otimes}B \to C$ such that $\varphi_A = \varphi \circ \iota_A$ and $\varphi_B = \varphi \circ \iota_B$.

Example 1.2.3. If $V = \bigoplus_{n \in \mathbb{Z}} V^{(n)}$ is any $\mathbb{Z}$-graded k-vector space, then writing

$$V^0 := \bigoplus_{n \in \mathbb{Z}} V^{(2n)} \text{ and } V^1 := \bigoplus_{n \in \mathbb{Z}} V^{(2n+1)}$$

gives V the structure of a super vector space. This gives a natural forgetful functor $\mathsf{GrVec}_k \to \mathsf{SVec}_k$.[2]

Example 1.2.4. In virtue of the previous example, associated to any (ungraded) vector space V, we have three superalgebras: the tensor algebra $\mathbb{T}V$, the symmetric algebra $\operatorname{Sym} V$, and the exterior algebra ΛV. Of these, only the last is supercommutative.

Example 1.2.5. If X is any topological space, then the cohomology algebra $\mathrm{H}^*(X, k)$ with coefficients in any field k equipped with the cup product is a supercommutative k-superalgebra.

Finally, we'll need the notion of the graded opposite. Recall that if A is an (ungraded) k-algebra, then the opposite A^{op} is defined to be the k-algebra with the same underlying vector space but multiplication given by $\mu^{\mathrm{op}} = \mu \circ s_{AA}$ where $s_{AA} : A \otimes A \to A \otimes A$ is the (ungraded) swap map. The same can now be done with superalgebras:

Definition 1.2.6. Let A be a k-superalgebra. Then the opposite superalgebra to A, denote $\hat{A}$ is the k-algebra with the same underlying super vector space structure but multiplication given by $\hat{\mu} = \mu \circ s_{AA}$, where $s_{AA} : A \otimes A \to A \otimes A$ is now the graded swap map, i.e. multiplication in $\hat{A}$ is defined for homogenous $x, y \in A$ by the rule

$$\hat{\mu}(x, y) := (-1)^{|x||y|} yx.$$

[2] As an aside, this amounts to the restriction of a $\mathbb{G}_{m,k}$-representation to the subgroup $\{\pm 1\} \subset \mathbb{G}_{m,k}$.

2 Clifford Algebras

2.1 Quadratic Spaces

As before, k is a field with $\operatorname{char} k \neq 2$.

Lemma/Definition 2.1.1. On a vector space V over k, the following pieces of data are equivalent:

(a) a symmetric k-bilinear form $\langle \cdot, \cdot \rangle$;
(b) a quadratic form on V, i.e. a map $q : V \to k$ that is
   i. homogenous of degree two, i.e. such that for all $v \in V$ and $t \in k$ we have $q(tv) = t^2 q(v)$, and
   ii. such that the polarization map

$$V \times V \to k, \quad (v, w) \mapsto q(v + w) - q(v) - q(w)$$

   is a k-bilinear form on V.

A k-vector space V with this additional structure is called a quadratic space over k, often denoted (V, q). Further, in this case:

(a) An element $v \in V$ is said to be isotropic if $q(v) = 0$.
(b) The quadratic space (V, q) is said to be nondegenerate if the corresponding bilinear form $\langle \cdot, \cdot \rangle$ is, which is to say that the linear map

$$V \to V^*, \quad v \mapsto \langle v, \cdot \rangle$$

   is isomorphism.

Proof. From $\langle \cdot, \cdot \rangle$, we get q as $q(v) := \langle v, v \rangle$, and from q we recover $\langle \cdot, \cdot \rangle$ by the parallelogram identity

$$\langle v, w \rangle = \frac{1}{2} \left(q(v + w) - q(v) - q(w) \right). \tag{1}$$

■

Definition 2.1.2. Let (V, q_V) and (W, q_W) be quadratic spaces over k. A morphism of quadratic spaces $T : (V, q_V) \to (W, q_W)$ is a k-linear map $T : V \to W$ that satisfies the following equivalent conditions:

(a) T preserves the symmetric bilinear forms on V and W, i.e. for all $v, v' \in V$, we have

$$\langle v, v' \rangle_V = \langle Tv, Tv' \rangle_W,$$

(b) T satisfies $T^* q_W = q_V$, which is to say that for all $v \in V$ we have

$$q_W(Tv) = q_V(v).$$

The equivalence of (a) and (b) is clear from the parallelogram identity (1) (recall that we're in $\operatorname{char} k \neq 2$). We denote by Quad_k the category of quadratic spaces over k, and $\mathsf{FDQuad}_k^{\mathsf{nd}}$ the category of finite dimensional nondegenerate quadratic spaces over k.

Remark 2. As is well-known, over fields k with $\operatorname{char} k = 2$, the notions of symmetric bilinear forms and quadratic forms are distinct. In this case, a quadratic space is defined to be a pair (V, q) of a vector space with a quadratic form on it.

Lemma 2.1.3. Let $(V, q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$, where $\operatorname{char} k \neq 2$. Then the quadratic form q on V can be diagonalized: there is a basis $v_1, \ldots, v_n$ of V (where $\dim V = n$) such that

$$\langle v_i, v_j \rangle = q(v_i)\delta_{ij}$$

for $1 \leq i, j \leq n$.

Proof. If $V \neq 0$, there is some $v \in V$ such that $q(v) \neq 0$ (else by the parallelogram identity, $\langle \cdot, \cdot \rangle \equiv 0$); define this to be v_1. Then it is easy to see that $\langle v \rangle^{\perp} \oplus \langle v \rangle \xrightarrow{\sim} V$, and that the restriction of q to $\langle v \rangle^{\perp}$ is nondegenerate as well, so we are done by induction. ∎

Definition 2.1.4.

(a) The automorphism group of the quadratic space (V, q) in Quad_k is called the orthogonal group of (V, q), denoted $\mathrm{O}(V, q)$, i.e.

$$\mathrm{O}(V, q) := \mathrm{Aut}_{\mathsf{Quad}_k}(V, q).$$

(b) When V is finite dimensional, we define the special orthogonal group $\mathrm{SO}(V, q)$ of (V, q) to be the subgroup of orientation-preserving automorphisms[3] of $\mathrm{O}(V, q)$:

$$\mathrm{SO}(V, q) := \mathrm{O}(V, q) \cap \mathrm{SL}(V).$$

Example 2.1.5. Let (V, q) be a quadratic space. For a non-isotropic $v \in V$, define the reflection in the hyperplane perpendicular to v to be the linear map $\rho_v : V \to V$ defined by

$$\rho_v(w) = w - 2\frac{\langle v, w \rangle}{\langle v, v \rangle} v,$$

for $w \in V$. Then the map $\rho_v \in \mathrm{O}(V, q) \smallsetminus \mathrm{SO}(V, q)$, and indeed $\det \rho_v = -1$.[4]

Finally, we will need:

Definition 2.1.6. For quadratic spaces (V, q_V) and (W, q_W), their orthogonal direct sum, denoted

$$(V \oplus W, q_V \oplus q_W)$$

is the quadratic space with underlying vector space $V \oplus W$ and quadratic form given by

$$(q_V \oplus q_W)(v, w) = q_V(v) + q_W(w) \tag{2}$$

for $v \in V$ and $w \in W$. This admits natural maps of quadratic spaces $(V, q_V), (W, q_W) \to (V \oplus W, q_V \oplus q_W)$ with the property (2), or equivalently that for all $v \in V$ and $w \in W$ that $\langle v, w \rangle = 0$, and is universal with respect to these properties.

[3] Note that you do not need to choose an orientation to talk about orientation-preserving automorphisms.

[4] Since $\rho_v(v) = -v$, if we complete v to a basis $v = v_1, v_2, \ldots, v_n$ for V, then we have

$$(\det \rho_v)(v_1 \wedge \cdots \wedge v_n) = -v \wedge \left(v_2 - 2\frac{\langle v, v_2 \rangle}{\langle v, v \rangle} v \right) \wedge \cdots \wedge \left(v_n - 2\frac{\langle v, v_n \rangle}{\langle v, v \rangle} v \right) = -v_1 \wedge \cdots \wedge v_n,$$

so that $\det \rho_v = -1$.

2.2 Clifford Algebras: Definition and the Canonical Filtration

In what follows, it will be helpful to have a slight generalization of the notion of a quadratic space.

Definition 2.2.1.

(a) A $\mathsf{generalized\ quadratic\ space}$ is a triple (V, A, q), where V is a vector space over k, and A is an associative k-algebra, and q is an A-valued quadratic form on V, i.e. a map $q : V \to A$ that is homogenous of degree two and such that the polarization map is an A-valued k-bilinear map on V. As before, this is equivalent to the data of an A-valued symmetric k-bilinear form on V.

(b) A morphism of generalized quadratic spaces $(V, A, q_V) \to (W, B, q_W)$ is the data of a linear map $T : V \to W$ and a k-algebra homomorphism $\varphi : A \to B$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{q_V} & A \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle \varphi} \\ W & \xrightarrow{q_W} & B \end{array}$$

commutes, i.e. such that for all $v \in V$ we have $q_W(Tv) = \varphi(q_V(v))$.

This gives us the category GQuad_k of generalized quadratic spaces over k.

Example 2.2.2. Any ordinary quadratic space (V, q) over k is of course a generalized quadratic space by taking $A = k$, and this gives us a fully faithful embedding of categories $\mathsf{Quad}_k \hookrightarrow \mathsf{GQuad}_k$.

Example 2.2.3. Suppose that A is a k-algebra. Then the squaring map $\mathrm{sq}_A : A \to A$ defined by

$$\mathrm{sq}_A(x) := x^2$$

for $x \in A$ is an A-valued quadratic form on the underlying vector space of A. Hence, there is a forgetful functor $F : \mathsf{Alg}_k \to \mathsf{GQuad}_k$ given by taking A to (A, A, sq_A). As we shall now see, this functor admits a left adjoint.

Definition 2.2.4. The $\mathsf{Clifford\ algebra\ functor}$ $\mathrm{Cl} : \mathsf{GQuad}_k \to \mathsf{Alg}_k$ is the left adjoint to the forgetful functor $F : \mathsf{Alg}_k \to \mathsf{GQuad}_k$. Explicitly, given a generalized quadratic space (V, A, q) over k, the Clifford algebra $\mathrm{Cl}(V, A, q)$ is an associative k-algebra along with a morphism of generalized quadratic spaces $\iota : (V, A, q) \to F(\mathrm{Cl}(V, A, q))$, i.e. a k-linear map $\iota : V \to \mathrm{Cl}(V, A, q)$ and a k-algebra homomorphism $\varphi : A \to \mathrm{Cl}(V, A, q)$ such that for all $v \in V$,

$$\iota(v)^2 = \varphi(q(v)), \tag{3}$$

which is universal with respect to this property and functorial in (V, A, q). Put another way, we have for any quadratic space (V, A, q) and associative k-algebra B a natural isomorphism

$$\mathrm{Hom}_{\mathsf{GQuad}_k}((V, A, q), F(B)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Alg}_k}(\mathrm{Cl}(V, A, q), B).$$

Clearly, if $\mathrm{Cl}(V, A, q)$ exists, it is unique up to unique isomorphism of k-algebras commuting with the morphism $\iota : (V, A, q) \to F(\mathrm{Cl}(V, A, q))$. One can construct $\mathrm{Cl}(V, A, q)$

explicitly as the quotient of $\mathbb{T}V *_k A$, the free k-product of the tensor algebra $\mathbb{T}V$ of V and A by the two-sided ideal generated by elements $(v \otimes v) * 1 - 1 * q(v)$ for $v \in V$, and this suffices to show existence.[5]

The restriction of Cl to the full subcategory Quad_k is still called the Clifford algebra functor $\mathrm{Cl} : \mathsf{Quad}_k \to \mathsf{Alg}_k$, and the Clifford algebra associated to a quadratic space (V, q) is denoted $\mathrm{Cl}(V, q)$. The above construction gives us a quotient map $\pi : \mathbb{T}V \twoheadrightarrow \mathrm{Cl}(V, q)$, which will be of use sometimes. We shall see in Theorem 2.2.6 below that the map ι in this case is an injection, so we shall henceforth suppress the notation ι and identify V with its image in $\mathrm{Cl}(V, q)$. The defining equation (3) implies that for $v, w \in V$, we have in $\mathrm{Cl}(V, q)$ that

$$v \cdot w + w \cdot v = 2\langle v, w\rangle, \tag{4}$$

and (4) implies (3) by taking $v = w$ (and using that $2 \in k^\times$).

Example 2.2.5. We have a functor $Z : \mathsf{Vec}_k \to \mathsf{Quad}_k$ given by taking a vector space V to the quadratic space $(V, 0)$ with zero quadratic form. The composite functor $\mathrm{Cl} \circ Z : \mathsf{Vec}_k \to \mathsf{Alg}_k$ is clearly isomorphic to the exterior algebra functor Λ.

The exterior algebra ΛV has more structure than just being an associative k-algebra: it is $\mathbb{Z}$-graded and filtered. We next investigate the existence of such additional structure on general Clifford algebras. Given the natural map $V \hookrightarrow \mathrm{Cl}(V, q)$, we get a filtration $F^r \mathrm{Cl}(V, q)$ on $\mathrm{Cl}(V, q)$ with

$$0 \subseteq F^0 \mathrm{Cl}(V, q) \subseteq F^1 \mathrm{Cl}(V, q) \subseteq \cdots,$$

where for $r \geq 0$, the subspace $F^r \mathrm{Cl}(V, q)$ is generated by elements of the form $v_1 \cdots v_s$ for $s \leq r$ and $v_0, \ldots, v_s \in V$. Clearly, this is the image of the usual filtration $F^\bullet \mathbb{T}V$ on $\mathbb{T}V$ under the quotient map $\pi : \mathbb{T}V \to \mathrm{Cl}(V, q)$. This filtration on $\mathrm{Cl}(V, q)$ gives the structure of a filtered k-algebra, and hence we may look at the associated graded k-algebra $\mathrm{Gr}\,\mathrm{Cl}(V, q)$.

Theorem 2.2.6. Let (V, q) be a quadratic space. The associated graded k-algebra of the Clifford algebra of (V, q) is naturally isomorphic to the exterior algebra of V, i.e. we have a functorial isomorphism of graded associative k-algebras

$$\Lambda V \cong \mathrm{Gr}\,\mathrm{Cl}(V, q).$$

Further:

(a) The above isomorphism commutes with the natural maps $V \xrightarrow{\sim} \Lambda^1 V$ and $V \xrightarrow{\iota} F^1 \mathrm{Cl}(V, q) \to \mathrm{Gr}^1 \mathrm{Cl}(V, q)$. In particular, the natural map $\iota : V \to \mathrm{Cl}(V, q)$ is injective.

(b) The Clifford algebra $\mathrm{Cl}(V, q)$ is a finite-dimensional k-algebra, and indeed

$$\dim \mathrm{Cl}(V, q) = 2^{\dim V}.$$

Proof. Since $\mathrm{Gr}\,\mathrm{Cl}(V, q)$ is an associative k-algebra with the property that the natural map $j : V \to \mathrm{Gr}\,\mathrm{Cl}(V, q)$ described above satisfies $j(v)^2 = 0$ for all $v \in V$, the universal

[5] As is, however, usual with adjoint functors of this sort, or equivalently universal properties, there is no one "correct" construction of the Clifford algebra–all constructions are equally valid.

property defining ΛV gives us a graded k-algebra homomorphism $\Lambda V \to \operatorname{Gr}\mathrm{Cl}(V,q)$. To go the other direction, we produce a Clifford action on ΛV. For any $v \in V$, we get two endomorphisms of ΛV: we have exterior multiplication by v, i.e. $v \wedge -$, which we denote by λ_v, and contraction with the covector $\langle v, \cdot\rangle$, which we denote by ι_v; this latter is the unique odd derivation of ΛV that satisfies

$$\iota_v(w) = \langle v, w\rangle$$

for $v, w \in V$. It is immediate to see that for any $v \in V$, we have $\lambda_v^2 = \iota_v^2 = 0$ and that

$$\lambda_v \cdot \iota_v + \iota_v \cdot \lambda_v = q(v) \cdot 1_{\Lambda V} \in \operatorname{End}(\Lambda V).$$

It follows that the linear map $V \to \operatorname{End}(\Lambda V)$ given by $v \mapsto \lambda_v + \iota_v$ extends to a k-algebra homomorphism $\eta : \mathrm{Cl}(V,q) \to \operatorname{End}(\Lambda V)$. The equation $(\lambda_v + \iota_v)(t) = tv$ for $v \in V$ and $t \in k = \Lambda^0 V$ shows that the map $V \to \operatorname{End}(\Lambda V)$, and hence $V \to \mathrm{Cl}(V,q)$ also, is injective. The map $\mathrm{Cl}(V,q) \to \Lambda V$ given by $\xi \mapsto \eta(\xi)(1)$ is then a k-algebra homomorphism which is easily seen to be filtered, and hence descends to a k-algebra morphism $\operatorname{Gr}\mathrm{Cl}(V,q) \to \operatorname{Gr}\Lambda V \cong \Lambda V$, which is the inverse of the above map. Now, the result in (a) is clear from the constructions, and the result in (b) follows from

$$\dim \mathrm{Cl}(V,q) = \dim \operatorname{Gr}\mathrm{Cl}(V,q) = \dim \Lambda V = 2^{\dim V}.$$

■

Remark 3. The above statement is still true in $\operatorname{char} k = 2$; for a proof, see [1, Prop. 1.2].

Remark 4. The linear isomorphism $\mathrm{Cl}(V,q) \to \Lambda V$ given by $\xi \mapsto \eta(\xi)(1)$ is often called the symbol map, and its inverse is called the quantization map. Note that these are not algebra isomorphisms. Given a quadratic space (V,q), we can define for $t \in k$ the quadratic form q_t on V to be $t \cdot q$, and this gives us a family of quadratic spaces (V, q_t), with the family of Clifford algebras $\mathrm{Cl}(V, q_t)$, which are all isomorphic to $\mathrm{Cl}(V,q)$ for $t \neq 0$ (when $k = \mathbb{C}$, say) degenerating at $t = 0$ to the exterior algebra $\mathrm{Cl}(V, q_0) = \mathrm{Cl}(V, 0) = \Lambda V$. Because of this, the Clifford algebra $\mathrm{Cl}(V,q)$ is often called a quantum deformation of the exterior algebra ΛV.

2.3 Clifford Algebras as Superalgebras

Any Clifford algebra $\mathrm{Cl}(V,q)$ has the structure of a k-superalgebra. There are several ways of thinking about this structure:

(a) Explicitly define $\mathrm{Cl}^0(V,q)$ to be the span of $v_1 \cdots v_{2r}$ for $v_i \in V$ and $r \geq 0$, and $\mathrm{Cl}^1(V,q)$ to be the span of $v_1 \cdots v_{2r+1}$ for $v_i \in V$ and $r \geq 0$. Equivalently, this is the superalgebra structure induced on $\mathrm{Cl}(V,q)$ by that on $\mathbb{T}V$ under the projection π because the ideal $\langle v \otimes v - q(v)\rangle_{v \in V}$ is generated by elements in $(\mathbb{T}V)^0$.

(b) By the functoriality of the Clifford algebra construction, we get for any quadratic space (V,q) a natural group homomorphism

$$\mathrm{O}(V,q) \to \operatorname{Aut}_{\mathsf{Alg}_k} \mathrm{Cl}(V,q).$$

In particular, the negation map $-1 \in \mathrm{O}(V,q)$ gives rise to a Clifford algebra automorphism $\alpha : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$ such that $\alpha^2 = 1_V$. This is a grading morphism.

Supercommutativity of this superalgebra is equivalent to $q = 0$, and hence Clifford algebras (other than exterior algebras) are not supercommutative in general. If $T : (V, q) \to (V', q')$ is a morphism of quadratic spaces, then the induced map of Clifford algebras $\mathrm{Cl}(T) : \mathrm{Cl}(V, q) \to \mathrm{Cl}(V', q')$ is clearly a morphism of k-superalgebras. In this way, the the Clifford algebra functor Cl can be lifted to a functor to SAlg_k. The importance of this additional structure comes from:

Lemma 2.3.1. For quadratic spaces (V, q_V) and (W, q_W), there is a natural isomorphism of k-superalgebras

$$\mathrm{Cl}(V \oplus W, q_V \oplus q_W) \to \mathrm{Cl}(V, q_V) \hat{\otimes} \mathrm{Cl}(W, q_W).$$

Proof. The algebra $\mathrm{Cl}(V, q_V) \hat{\otimes} \mathrm{Cl}(W, q_W)$ is an associative k-algebra and the linear map

$$V \oplus W \to \mathrm{Cl}(V, q_V) \hat{\otimes} \mathrm{Cl}(W, q_W), \quad (v, w) \mapsto v \otimes 1 + 1 \otimes w$$

has the property that

$$(v \otimes 1 + 1 \otimes w)^2 = (q_V \oplus q_W)(v, w),$$

where cross terms cancel by the definition of the super tensor product on the right side. Therefore, by the universal property, there is a k-algebra morphism

$$\mathrm{Cl}(V \oplus W, q_V \oplus q_W) \to \mathrm{Cl}(V, q_V) \hat{\otimes} \mathrm{Cl}(W, q_W). \tag{5}$$

extending this linear map. On the other hand, the inclusions

$$(V, q_V), (W, q_W) \hookrightarrow (V \oplus W, q_V \oplus q_W)$$

induce k-superalgebra morphisms

$$\mathrm{Cl}(V, q_V), \mathrm{Cl}(W, q_W) \hookrightarrow \mathrm{Cl}(V \oplus W, q_V \oplus q_W)$$

with the property that for $\xi \in \mathrm{Cl}(V, q_V)$ and $\zeta \in \mathrm{Cl}(W, q_W)$ the supercommutator between ξ and ζ vanishes, i.e.

$$[\xi, \zeta] = 0 \in \mathrm{Cl}(V \oplus W, q_V \oplus q_W).$$

To see why this is true, note that it hoods for $\xi \in V$ and $\zeta \in W$ from Equation (4), and follows in general from that special case and the fact that V (resp. W) generates $\mathrm{Cl}(V, q_V)$ (resp. $\mathrm{Cl}(W, q_W)$) as a k-algebra. It follows from Lemma 1.2.2 that there is a unique k-superalgebra morphism

$$\mathrm{Cl}(V, q_V) \hat{\otimes} \mathrm{Cl}(W, q_W) \to \mathrm{Cl}(V \oplus W, q_V \oplus q_W)$$

that commutes with the maps $\iota_{\mathrm{Cl}(V, q_V)}$ and $\iota_{\mathrm{Cl}(W, q_W)}$, and this is the inverse to the morphism (5). ∎

Remark 5. The opposite superalgebra to a Clifford algebra of a quadratic space (V, q) is that of $(V, q_{-1}) := (V, -q)$, i.e.

$$\widehat{\mathrm{Cl}(V, q)} = \mathrm{Cl}(V, -q).$$

Finally, we'll need:

Lemma 2.3.2. Let $(V, q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$. If $\xi \in \mathrm{Cl}(V, q)$ satisfies $[\xi, v] = 0$ for all $v \in V$, then $\xi \in k$. In particular, the center $Z(\mathrm{Cl}(V, q)) = k$.

Proof. Writing $\xi = \xi^0 + \xi^1$ for $\xi^i \in \mathrm{Cl}^i(V, q)$, the condition $[\xi, v] = 0$ says that

$$\xi^0 v = v\xi^0 \text{ and } \xi^1 v = -v\xi^1 \text{ for all } v \in V. \tag{6}$$

By Lemma 2.1.3, we can produce a basis $v_1, \ldots, v_n$ of V such that $q(v_i) \neq 0$ for $i = 1, \ldots, n$ but $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Using this basis, we can write $\xi^0 = \eta^0 + v_1\eta^1$ where the even η^0 and odd η^1 only involve $v_2, \ldots, v_n$. In particular, by orthogonality, we have

$$[v_1, \eta^0] = [v_1, \eta^1] = 0.$$

Setting $v = v_1$ in (6) gives us and using this supercommutativity, we get

$$v_1\eta^0 - v_1^2\eta^1 = \eta^0 v_1 + v_1\eta^1 v_1 = (\eta^0 + v_1\eta^1)v_1 = v_1(\eta^0 + v_1\eta^1) = v_1\eta^0 + v_1^2\eta^1$$

so that

$$\eta^1 = q(v_1)^{-1}v_1^2\eta^1 = 0.$$

Therefore, $\xi^0 = \eta^0$ doesn't involve v_1. Proceeding inductively, we conclude that ξ^0 doesn't involve any v_j, and hence $\xi^0 \in k$. Similarly, we write $\xi^1 = \zeta^1 + v_1\zeta^0$ for odd ζ^1 and even ζ^0 involving only $v_2, \ldots, v_n$, and from this conclude as before that $\zeta^0 = 0$. This shows that ξ^1 doesn't involve v_1, and, as before, inductively, that ξ^1 doesn't involve any v_j. But ξ^1 is odd, so the only way this can happen is if $\xi^1 = 0$. This tells us that $\xi \in k$ as needed. ■

Remark 6. This result is not true when the quadratic space (V, q) is not nondegenerate. For instance, if $q = 0$ so $\mathrm{Cl}(V, q) = \Lambda V$, then for any $u, v \in V$, the element $\xi := 1 + uv \in \Lambda V$ satisfies for any $w \in V$ that

$$[1 + uv, w] = 0.$$

2.4 Examples: $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_n^{\mathbb{C}}$

Let $k = \mathbb{R}$. The nondegenerate quadratic forms, or equivalently nondegenerate bilinear forms on finite-dimensional $\mathbb{R}$-vector spaces are classified completely by their signature (p, q), where p, where $p \geq 0$ (resp. $q \geq 0$) is the maximal dimension of a subspace to which the restriction of the form is negative definite (resp. positive definite). For any pair (p, q) of nonnegative integers, we consider the vector space $\mathbb{R}^{p+q}$ with quadratic form

$$q\left(\sum_{i=0}^{p+q} x^i \mathrm{e}_i\right) = -\sum_{i=0}^{p}(x^i)^2 + \sum_{i=p+1}^{p+q}(x^i)^2,$$

and denote the corresponding Clifford algebra by $\mathrm{Cl}_{p,q}$. These are, up to isomorphism, the only finite-dimensional Clifford algebras coming from nondegenerate real quadratic spaces.[6]

[6] If a quadratic form over $\mathbb{R}$ is degenerate, then we can split off an orthogonal summand where it is zero, and we know by Lemma 2.3.1 what the structure of the resulting Clifford algebra looks like as well.

Similarly, if $k = \mathbb{C}$, then all nondegenerate quadratic forms on finite-dimensional $\mathbb{C}$-vector spaces of the same dimension are isomorphic; for concreteness, we choose for $n \geq 0$ the complex vector space $\mathbb{C}^n$ with quadratic form

$$q\left(\sum_{i=0}^{n} x^i \mathrm{e}_i\right) = -\sum_{i=0}^{n} (x^i)^2$$

and denote the result Clifford algebra by $\mathrm{Cl}_n^{\mathbb{C}}$. As before, these are the only complex Clifford algebras coming from nondegenerate complex quadratic spaces.

In low dimensions, we can recognize these explicitly: if for integer $n \geq 1$ we let $k(n)$ denote the k-algebra of $n \times n$ matrices with values in k, then we have the isomorphisms of ungraded real algebras

$$\begin{aligned}
\mathrm{Cl}_{1,0} &\cong \mathbb{R}[x]/(x^2+1) && \cong \mathbb{C}, \\
\mathrm{Cl}_{0,1} &\cong \mathbb{R}[x]/(x^2-1) && \cong \mathbb{R} \oplus \mathbb{R}, \\
\mathrm{Cl}_{2,0} &\cong \mathbb{R}\{i,j\}/(i^2+1, j^2+1, ij+ji) && \cong \mathbb{H}, \\
\mathrm{Cl}_{0,2} &\cong \mathbb{R}\{i,j\}/(i^2-1, j^2-1, ij+ji) && \cong \mathbb{R}(2), \text{ and} \\
\mathrm{Cl}_{1,1} &\cong \mathbb{R}\{i,k\}/(i^2-1, k^2+1, ik+ki) && \cong \mathbb{R}(2),
\end{aligned}$$

where $\{\cdot\}$ denotes the free associative algebra generated by the variables inside, and the last two isomorphisms are given by

$$i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and } k = ij = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Similarly, we have the isomorphisms of ungraded complex algebras

$$\begin{aligned}
\mathrm{Cl}_1^{\mathbb{C}} &\cong \mathbb{C}[x]/(x^2+1) && \cong \mathbb{C} \oplus \mathbb{C}, \text{ and} \\
\mathrm{Cl}_2^{\mathbb{C}} &\cong \mathbb{C}\{i,j\}/(i^2-1, j^2-1, ij+ji) && \cong \mathbb{C}(2).
\end{aligned}$$

3 Pin and Spin Groups

3.1 The Twisted Adjoint Action and the Clifford-Lipschitz Group

We next define an important action of the group of units $\mathrm{Cl}^\times(V,q)$ on $\mathrm{Cl}(V,q)$. Recall from §2.3 that we denote by α the grading morphism of the superalgebra $\mathrm{Cl}(V,q)$.

Definition 3.1.1. Given a Clifford algebra $\mathrm{Cl}(V,q)$, the twisted adjoint action of the group of units $\mathrm{Cl}^\times(V,q) \subset \mathrm{Cl}(V,q)$ on $\mathrm{Cl}(V,q)$ is the group homomorphism

$$\mathrm{Ad} : \mathrm{Cl}^\times(V,q) \to \mathrm{Aut}\,\mathrm{Cl}(V,q)$$

defined by

$$\mathrm{Ad}_\xi(\zeta) = \alpha(\xi) \cdot \zeta \cdot \xi^{-1}.$$

The importance of this action comes from the following lemma:

Lemma 3.1.2. Let (V,q) be any quadratic space. Then a vector $v \in V$ is not isotropic iff it is invertible in $\mathrm{Cl}(V,q)$, in which case we have $v^{-1} = q(v)^{-1}v$. In this case, we have for any other $w \in V$ that

$$\mathrm{Ad}_v(w) = \rho_v(w).$$

In particular, we have for any $v \in V$ with $q(v) \neq 0$ that $\mathrm{Ad}_v(V) \subset V$.

In English, the twisted adjoint action by a vector $v \in V$ on another vector $w \in V$ is the same as reflection in the hyperplane perpendicular to v.

Proof. The first statement is clear, and for the second we have

$$\mathrm{Ad}_v(w) = \alpha(v) \cdot w \cdot v^{-1} = -v \cdot w \cdot \frac{1}{q(v)} v = -\frac{1}{q(v)}(v \cdot w \cdot v).$$

Now using (4), we have $w \cdot v = -v \cdot w + 2\langle v, w\rangle$, and therefore

$$\mathrm{Ad}_v(w) = -\frac{1}{q(v)}\left(-v^2 \cdot w + 2\langle v,w\rangle v\right) = w - 2\frac{\langle v,w\rangle}{\langle v,v\rangle} v = \rho_v(w).$$

∎

This motivates the following definition:

Definition 3.1.3. For a quadratic space (V,q), we define the Clifford-Lipschitz group[7] $\Gamma(V,q)$ of (V,q) to be the subgroup of $\mathrm{Cl}^\times(V,q)$ that preserves V under the twisted adjoint representation:

$$\Gamma(V,q) := \{\xi \in \mathrm{Cl}^\times(V,q) : \mathrm{Ad}_\xi(V) \subset V\}.$$

Finally, we define the even Clifford-Lipschitz group $\Gamma^0(V,q)$ by

$$\Gamma^0(V,q) := \Gamma(V,q) \cap \mathrm{Cl}^0(V,q).$$

[7] This group is often called the Clifford group of the quadratic space (V,q); for instance, this is the terminology used in the seminal paper [2] by Atiyah, Bott, and Shapiro. However, it seems that this notion was never mentioned or used by William Kingdon Clifford himself, but rather first discovered and used by Rudolf Lipschitz in 1880/86; see [3, Ch. 17] and the references at the end of that chapter.

The previous lemma says that any nonisotropic vector belongs to the Clifford-Lipschitz group. By definition, the twisted adjoint action gives rise to a representation of $\Gamma(V,q)$ on V; we let $\rho : \Gamma(V,q) \to \mathrm{GL}(V)$ denote this representation. By the computation in the previous lemma, this agrees with the notation previously established for reflections on $\xi = v \in V$ with $q(v) \neq 0$. Note that we have a natural inclusion $k^\times \hookrightarrow \Gamma(V,q)$ as constants, and all these constants act trivially under the twisted adjoint action. The first result we need is that these are all the elements of the kernel in the nondegenerate case:

Lemma 3.1.4. If $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$, then the sequence of groups

$$1 \to k^\times \to \Gamma(V,q) \xrightarrow{\rho} \mathrm{GL}(V)$$

is exact.

Proof. If $\xi \in \ker \rho$, then for all $v \in V$ we have $\alpha(\xi)v = v\xi$, and this says exactly that for all $v \in V$ we have $[\xi, v] = 0$. Hence, $\xi \in k$ by Lemma 2.3.2, so $\xi \in \mathrm{Cl}^\times(V,q) \cap k = k^\times$. ■

Next, we find the image of ρ.

3.2 The Spinor Norm and the Fundamental Exact Sequence

Definition 3.2.1. Let (V,q) be a quadratic space.

(a) The ungraded opposite $\mathrm{Cl}(V,q)^{\mathrm{op}}$ to the Clifford algebra $\mathrm{Cl}(V,q)$ is an associative k-algebra that has a map $V \to \mathrm{Cl}(V,q)^{\mathrm{op}}$ such that $v^2 = q(v) \in \mathrm{Cl}(V,q)^{\mathrm{op}}$, and hence by the universal property extends to a k-algebra homomorphism

$$\cdot^{\mathrm{t}} : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)^{\mathrm{op}}.$$

This is called the transpose anti-involution, and is given explicitly for $r \geq 0$ and $v_1, \ldots, v_r \in V$ by

$$(v_1 \ldots v_r)^{\mathrm{t}} = v_r \cdots v_1.$$

(b) The Clifford conjugation map is the anti-involution $\bar{\cdot} : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$ defined by

$$\overline{\xi} := \alpha(\xi^{\mathrm{t}}).$$

The spinor norm is the map

$$\mathrm{N} : \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q), \quad \xi \mapsto \xi \cdot \overline{\xi}.$$

Remark 7. For $\lambda \in k \subset \mathrm{Cl}(V,q)$, we have $\alpha(\lambda) = \lambda^{\mathrm{t}} = \overline{\lambda} = \lambda$, and hence $\mathrm{N}(\lambda) = \lambda^2$. For $v \in V \subset \mathrm{Cl}(V,q)$, we have $v^{\mathrm{t}} = v$, but $\alpha(v) = -v$, so that $\overline{\lambda} = -v$ and hence $\mathrm{N}(v) = -v^2 = -q(v)$. Clifford conjugation on $\mathrm{Cl}_{1,0} \cong \mathbb{C}$ is just complex conjugation, and the spinor norm the usual complex norm. Similarly, Clifford conjugation on $\mathrm{Cl}_{2,0} \cong \mathbb{H}$ is quaternionic conjugation, and the spinor norm is the quaternionic norm.

Lemma 3.2.2. Let $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$.

(a) If $\xi \in \Gamma(V,q)$, then $\mathrm{N}(\xi) \in k^\times$.

(b) If $\eta \in \mathrm{Cl}(V,q)$ is such that $\mathrm{N}(\eta) \in k$, then for all $\xi \in \mathrm{Cl}(V,q)$ we have

$$\mathrm{N}(\xi\eta) = \mathrm{N}(\xi)\,\mathrm{N}(\eta).$$

In particular, the map

$$\mathrm{N} : \Gamma(V,q) \to k^\times$$

defined by (a) is a group homomorphism with the property that $\mathrm{N}(\alpha(\xi)) = \mathrm{N}(\xi)$ for all $\xi \in \Gamma(V,q)$.

(c) The image of $\rho : \Gamma(V,q) \to \mathrm{GL}(V)$ is contained in $\mathrm{O}(V,q)$, i.e. $\Gamma(V,q)$ acts by isometries on (V,q).

Proof.

(a) Note that for $\xi \in \mathrm{Cl}^\times(V,q)$ and $\eta \in \mathrm{Cl}(V,q)$, we have that

$$\mathrm{Ad}_{\overline{\xi}}(\eta) = \left(\mathrm{Ad}_{\xi^{-1}}(\eta^{\mathrm{t}})\right)^{\mathrm{t}}. \tag{7}$$

In particular, if $\xi \in \Gamma(V,q)$, then also $\overline{\xi} \in \Gamma(V,q)$ with $\rho_{\overline{\xi}} = \rho_{\xi^{-1}}$, and hence also $\mathrm{N}(\xi) \in \Gamma(V,q)$. Therefore, by Lemma 3.1.4, it suffices to show that $\mathrm{N}(\xi) \in \ker\rho$, and this follows from

$$\rho_{\mathrm{N}(\xi)} = \rho_{\xi\cdot\overline{\xi}} = \rho_\xi \circ \rho_{\overline{\xi}} = \rho_\xi \circ \rho_{\xi^{-1}} = \rho_{\xi\cdot\xi^{-1}} = \rho_1 = 1_V.$$

(b) For $\xi \in \mathrm{Cl}(V,q)$ and $\eta \in \mathrm{Cl}(V,q)$ such that $\mathrm{N}(\eta) \in k$, we have

$$\mathrm{N}(\xi\eta) = \xi \cdot \eta \cdot \overline{\eta} \cdot \overline{\xi} = \xi\,\mathrm{N}(\eta)\overline{\xi} = \mathrm{N}(\xi)\,\mathrm{N}(\eta).$$

Finally, we have

$$\mathrm{N}(\alpha(\xi)) = \alpha(\xi)\overline{\alpha(\xi)} = \alpha(\xi)\xi^t = \alpha(\xi \cdot \alpha(\xi^{\mathrm{t}})) = \alpha(\mathrm{N}(\xi)) = \mathrm{N}(\xi),$$

where in the last step we have used (a).

(c) We have for $v \in V$ and $\xi \in \Gamma(V,q)$ that

$$q(\rho_\xi v) = -\mathrm{N}(\rho_\xi v) = -\mathrm{N}(\alpha(\xi)v\xi^{-1}).$$

By using that that $\mathrm{N}(v), \mathrm{N}(\xi) \in k$, we have by (b) that

$$-\mathrm{N}(\alpha(\xi)v\xi^{-1}) = -\mathrm{N}(\alpha(\xi))\,\mathrm{N}(v)\,\mathrm{N}(\xi^{-1}) = -\mathrm{N}(\xi)\,\mathrm{N}(v)\,\mathrm{N}(\xi)^{-1} = q(v).$$

■

This result gives us a map $\rho : \Gamma(V,q) \to \mathrm{O}(V,q)$. In fact, this map is surjective:

Theorem 3.2.3 (E. Cartan-Dieoudonné)**.** If $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$ with $\operatorname{char} k \neq 2$ and $\dim V = n \geq 1$, then every element of $\mathrm{O}(V,q)$ can be written as a product of n or fewer reflections ρ_v for non-isotropic V.

Proof. See [4, Thm. 6.6] or [5, Thm. 1.1]. We only remark here that this uses $\operatorname{char} k \neq 2$ crucially, and there is a(n) (essentially unique) counterexample to this result in $\operatorname{char} k = 2$, see [4, Ch. 14]. ■

From Theorem 3.2.3 and Lemma 3.1.4, for each $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$, we get the fundamental exact sequence

$$1 \to k^\times \to \Gamma(V,q) \xrightarrow{\rho} \mathrm{O}(V,q) \to 1.$$

From this sequence , we also obtain:

Corollary 3.2.4. If $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$, then the Clifford-Lipschitz group $\Gamma(V,q)$ is generated by non-isotropic $v \in V$. In particular,

$$\rho\left(\Gamma^0(V,q)\right) \subset \mathrm{SO}(V,q).$$

Proof. For any $\xi \in \Gamma(V,q)$ by Theorem 3.2.3 we can write $\rho_\xi = \rho_{v_1} \circ \cdots \circ \rho_{v_r}$ for some $r \geq 0$ and non-isotropic $v_1, \ldots, v_r \in V$. Then, we have $\xi^{-1}v_1 \cdots v_r \in \ker\rho$, and hence that $\xi^{-1}v_1 \cdots v_r = \lambda$ for some $\lambda \in k^\times$. Therefore, we get that $\xi = (\lambda^{-1}v_1)v_2 \cdots v_r$ as needed. For the second part, if we write $\xi \in \Gamma^0(V,q)$ as $\xi = v_1 \cdots v_r$ for some $r \geq 0$ and non-istropic $v_1, \ldots, v_r \in V$, then $\xi \in \mathrm{Cl}^0(V,q)$ implies that r is even, and hence by Example 2.1.5, it follows that $\rho_\xi = \rho_{v_1} \circ \cdots \circ \rho_{v_r} \in \mathrm{SO}(V,q)$ as needed. ■

3.3 Pin and Spin Groups

Definition 3.3.1. Let (V,q) be a quadratic space.

(a) The pin group $\mathrm{Pin}(V,q)$ of (V,q) is the subgroup of the Clifford-Lipschitz group $\Gamma(V,q)$ consisting of elements with spinor norm ± 1, i.e.

$$\mathrm{Pin}(V,q) := \{\xi \in \Gamma(V,q) : \mathrm{N}(\xi) = \pm 1\}.$$

(b) The spin group $\mathrm{Spin}(V,q)$ of (V,q) is the subgroup of thepPin group $\mathrm{Pin}(V,q)$ consisting of even elements, i.e.

$$\mathrm{Spin}(V,q) := \mathrm{Pin}(V,q) \cap \mathrm{Cl}^0(V,q).$$

Definition 3.3.2. Suppose $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$. Then morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & k^\times \arrow[r] \arrow[d, "\mathrm{sq}_k"] & \Gamma(V,q) \arrow[r] \arrow[d, "\mathrm{N}"] & \mathrm{O}(V,q) \arrow[r] \arrow[d, dashed, "\mathrm{N}"] & 1 \\
1 \arrow[r] & (k^\times)^2 \arrow[r] & k^\times \arrow[r] & k^\times/(k^\times)^2 \arrow[r] & 1
\end{tikzcd}$$

defines a morphism $\mathrm{N} : \mathrm{O}(V,q) \to k^\times/(k^\times)^2$ also called the spinor norm.

We define for $n \geq 1$, the group $\mu_n(k) := \{\lambda \in k^\times : \lambda^n = 1\}$, so that $\mathrm{Pin}(V,q) := \mathrm{N}^{-1}(\mu_2(k))$. By Corollary 3.2.4, the map ρ takes $\mathrm{Spin}(V,q)$ to $\mathrm{SO}(V,q)$. From the fundamental exact sequence and the definition of the spinor norm, we obtain immediately for $(V,q) \in \mathsf{FDQuad}_k^{\mathsf{nd}}$ the spinor exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mu_4(k) \arrow[r] & \mathrm{Pin}(V,q) \arrow[r, "\rho"] & \mathrm{O}(V,q) \arrow[r, "{[\mathrm{N}]}"] & k^\times/(\mu_2(k)(k^\times)^2) \\
1 \arrow[r] & \mu_4(k) \arrow[r] \arrow[u, equal] & \mathrm{Spin}(V,q) \arrow[r, "\rho"] \arrow[u, hook] & \mathrm{SO}(V,q) \arrow[r, "{[\mathrm{N}]}"] \arrow[u, hook] & k^\times/(\mu_2(k)(k^\times)^2). \arrow[u, equal]
\end{tikzcd}$$

In [1], the authors call a field k a **spin field** if $k^\times/(\mu_2(k)(k^\times)^2) = \{1\}$; examples of spin fields include all algebraically closed fields, and some others such as $k = \mathbb{R}$ or $k = \mathbb{F}_p$ for $p \equiv 3 \pmod 4$. For spin fields, the maps ρ above are therefore surjections, and the spinor exact sequences become

$$\begin{tikzcd}
1 \arrow[r] & \mu_4(k) \arrow[r] & \mathrm{Pin}(V,q) \arrow[r, "\rho"] & \mathrm{O}(V,q) \arrow[r] & 1 \\
1 \arrow[r] & \mu_4(k) \arrow[r] \arrow[u, equal] & \mathrm{Spin}(V,q) \arrow[r, "\rho"] \arrow[u, hook] & \mathrm{SO}(V,q) \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

If $k = \mathbb{R}$ or $k = \mathbb{C}$, then in the sequence of subgroups

$$\mathrm{Spin}(V,q) \subset \mathrm{Pin}(V,q) \subset \Gamma(V,q) \subset \mathrm{Cl}^\times(V,q),$$

each subgroup is closed in the next one, and the last one is an open subgroup of $\mathrm{Cl}(V,q)$, and hence, by Cartan's Theorem, this is a sequence of closed (real) Lie subgroups. The spin groups associated to the Clifford algebras $\mathrm{Cl}_{p,q}$ are denoted by $\mathrm{Spin}_{p,q}$, and those to $\mathrm{Cl}_n^{\mathbb{C}}$ are denoted by $\mathrm{Spin}_n(\mathbb{C})$. Note the special case of $p = n$ and $q = 0$, where $\mathrm{SO}(V,q) = \mathrm{SO}(V,-q) = \mathrm{SO}_n$. Recall that $\pi_0 \mathrm{SO}_n = 0$ for $n \geq 0$ and

$$\pi_1 \mathrm{SO}_n = \begin{cases} 0, & n = 1, \\ \mathbb{Z}, & n = 2, \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

One main result of this section is:

Theorem 3.3.3. We have the short exact sequence of Lie groups

$$1 \to \{\pm 1\} \to \mathrm{Spin}_n \to \mathrm{SO}_n \to 1.$$

For $n \geq 2$, the cover $\mathrm{Spin}_n \to \mathrm{SO}_n$ is nontrivial: it is the unique connected degree two cover of SO_n. In particular, for $n \geq 3$, the map $\mathrm{Spin}_n \to \mathrm{SO}_n$ is the universal cover of the group SO_n.

Proof. All that remains to show is that for $n \geq 2$, the points $\pm 1 \in \mathrm{Spin}_n$ can be connected by a path in Spin_n. For this use Lemma 2.1.3 to pick $e, f \in \mathbb{R}^n$ orthogonal with $q(e) = q(f) = -1$. Then the path

$$\gamma(t) = (e \cos t + f \sin t)(e \cos(\pi - t) + f \sin(\pi - t)) = \cos 2t + ef \sin 2t$$

lies in Spin_n and satisfies $\gamma(0) = 1$ and $\gamma(\pi/2) = -1$. ∎

Remark 8. In closing, we remark that there seems to be some disagreement about the definition of the (s)pin groups. We have followed the convention in [1, Ch. 1]. The groups in this convention have the property that over $k = \mathbb{C}$, in the case of the algebra $\mathrm{Cl}_n^{\mathbb{C}}$, we have the exact sequence

$$1 \to \mu_4(\mathbb{C}) = \{\pm 1, \pm \mathrm{i}\} \to \mathrm{Spin}_n(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C}) \to 1,$$

so that $\mathrm{Spin}_n(\mathbb{C})$ is a degree 4 cover of $\mathrm{SO}_n(\mathbb{C})$. On the other hand, in [2], Atiyah-Bott-Shapiro define the pin group to be the kernel of the norm homomorphism. In the negative definite case over $k = \mathbb{R}$ (the only case that Atiyah-Bott-Shapiro consider), these conventions yield the same groups, but in general these notions are distinct. To make the connection between these conventions precise, let us temporarily use the notation

$$\mathrm{Pin}^+(V,q) := \ker N = \{\xi \in \Gamma(V,q) : \mathrm{N}(\xi) = 1\} \text{ and } \mathrm{Spin}^+(V,q) := \mathrm{Pin}^+(V,q) \cap \mathrm{Spin}(V,q).$$

The version of the spinor exact sequences for these groups is

$$\begin{tikzcd}
1 \arrow[r] & \mu_2(k) \arrow[r] \arrow[d, equal] & \mathrm{Pin}^+(V,q) \arrow[r, "\rho"] & \mathrm{O}(V,q) \arrow[r, "\mathrm{N}"] & k^\times/(k^\times)^2 \arrow[d, equal] \\
1 \arrow[r] & \mu_2(k) \arrow[r] & \mathrm{Spin}^+(V,q) \arrow[r, "\rho"] \arrow[u, hook] & \mathrm{SO}(V,q) \arrow[r, "\mathrm{N}"] \arrow[u, hook] & k^\times/(k^\times)^2,
\end{tikzcd}$$

and as before, these sequences are not exact on the right in general. Let us consider two cases in detail:

(a) The case of a definite form over $k = \mathbb{C}$. In this case, these sequences become

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, equal] & \mathrm{Pin}_n^+(\mathbb{C}) \arrow[r, "\rho"] & \mathrm{O}_n(\mathbb{C}) \arrow[r] & 1 \\
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{Spin}_n^+(\mathbb{C}) \arrow[r, "\rho"] \arrow[u, hook] & \mathrm{SO}_n(\mathbb{C}) \arrow[r] \arrow[u, hook] & 1,
\end{tikzcd}$$

so that $\mathrm{Spin}_n^+(\mathbb{C})$ is a double cover of $\mathrm{SO}_n(\mathbb{C})$. A similar reasoning as in the proof of Theorem 3.3.3 shows that $\mathrm{Spin}_n^+(\mathbb{C})$ is connected, so that $\mathrm{Spin}_n^+(\mathbb{C}) \subset \mathrm{Spin}_n(\mathbb{C})$ is an index-2 subgroup and $\mathrm{Spin}_n^+(\mathbb{C}) \xrightarrow{\rho} \mathrm{SO}_n(\mathbb{C})$ is a nontrivial double cover.

(b) The case of a negative definite form over $k = \mathbb{R}$. In this case, these sequences become

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, equal] & \mathrm{Pin}_n^+ \arrow[r, "\rho"] & \mathrm{O}_n \arrow[r, "\mathrm{N}"] & \{\pm 1\} \arrow[d, equal] \\
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{Spin}_n^+ \arrow[r, "\rho"] \arrow[u, hook] & \mathrm{SO}_n \arrow[r, "\mathrm{N}"] \arrow[u, hook] & \{\pm 1\}.
\end{tikzcd}$$

Note that these sequences are not necessarily exact at the rightmost stage, and indeed, in this case, we have $\mathrm{N}(\mathrm{O}_n) = \mathrm{N}(\mathrm{SO}_n) = 1$ because O_n can be generated by reflections in vectors of unit length. Therefore, in this case, we have

$$\mathrm{Pin}_n^+ = \mathrm{Pin}_n \text{ and } \mathrm{Spin}_n^+ = \mathrm{Spin}_n$$

as promised.

References

[1] J. H. Blaine Lawson and M.-L. Michelsohn, *Spin Geometry*, vol. 38 of *Princeton Mathematical Series*. Princeton University Press, 1989.

[2] M. Atiyah, R. Bott, and A. Shapiro, "Clifford Modules," *Topology (Oxford)*, vol. 3, pp. 3–38, 1964.

[3] P. Lounesto, *Clifford Algebras and Spinors*, vol. 286 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, second ed., 2001.

[4] L. C. Grove, *Classical Groups and Geometric Algebra*, vol. 39 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001.

[5] E. Meinreken, *Clifford Algebras and Lie Theory*, vol. 58 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2013.