Elliptic K3 Surfaces

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Abstract

This paper has been written in fulfillment of the requirements of the class Math 277Z: K3 Surfaces taught at Harvard by Dori Bejleri in the Spring 2023 semester. After developing the fundamentals of the theory of K3 surfaces, this paper proves that a given complex algebraic K3 surface is elliptic iff it admits a polarization by the hyperbolic plane.

Contents

1 Preliminaries

Throughout, we will closely follow [1] and [2], referring only rarely as necessary to [3].

1.1 Preliminaries on Surfaces

Throughout, we let k be a fixed algebraically closed base field. A variety X over k is defined to be a separated, geometrically integral scheme of finite type over k. For a smooth variety X, we write ω_X for the canonical sheaf $\det \Omega_X$ of X and K_X for its class in $\operatorname{Pic} X$.

Next, suppose that X is a smooth projective surface. Let $\operatorname{Div} X$ be the group of Weil divisors on X, i.e. the free abelian group generated by irreducible codimension 1 subvarieties of X. Recall that two divisors $C, C' \in \operatorname{Div} X$ are said to be linearly equivalent if there is a function $f \in k(X)$ such that $C = C' + \operatorname{div}(f)$. The group of Weil divisors modulo linear equivalence is identified naturally with the group of line bundles $\operatorname{Pic} X$ on X as usual. The intersection pairing on $\operatorname{Div} X$ and $\operatorname{Pic} X$ is defined by the following theorem:

Theorem 1.1.1. There is a unique symmetric $\mathbb{Z}$-bilinear pairing

$$\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$$

denoted by $(L, L') \mapsto L \cdot L'$ satisfying the following property: if $C, D \in \operatorname{Div} X$ are smooth curves that intersect transversally, then $[C] \cdot [D] = \#(C \cap D)$, the number of points of intersection of C and D.

Proof. See [4, V.1.6]. ∎

The lift of this pairing to $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ is usually denoted by $(C, D) \mapsto (C, D)_X$, although we will abuse notations and freely move between the two alternatives. There are two further notions of equivalence on $\operatorname{Div} X$:

(a) Divisors $C, C' \in \operatorname{Div} X$ are said to be algebraically equivalent if there is a connected curve T, two closed points $0, 1 \in T$ and a divisor E on $X \times T$, flat over T, such that $E|_{X\times 0} - E|_{X\times 1} = C - C'$.
(b) Divisors $C, C' \in \operatorname{Div} X$ are said to be numerically equivalent if for every other $D \in \operatorname{Div} X$, we have $(C, D)_X = (C', D)_X$.

These relations satisfy

$$\text{linear equivalence} \Rightarrow \text{algebraic equivalence} \Rightarrow \text{numerical equivalence}.$$

For the first implication, take $T = \mathbb{P}^1 = \operatorname{Proj} k[s, t]$ and $E = \operatorname{div}(sf - t)$ in $X \times \mathbb{P}^1$. For the second implication, let H be a very ample divisor on X and use it to embed $X \hookrightarrow \mathbb{P}^n$ for some n. This allows us to embed $X \times T$ and so E into $\mathbb{P}^n \times T = \mathbb{P}^n_T$. By flatness of $E \to T$ and connectedness of T, the Hilbert polynomials of the fibers $E \to T$ above closed points are constant. In particular, the fibers have the same degree. On the other hand, the degree of the fiber over any closed point $t \in T$ is exactly $(E|_{X\times t}, H)$, since H corresponds to the hyperplane class in $\mathbb{P}^n$. It follows that

$$(C, H)_X - (C', H)_X = (E|_{X\times 0}, H)_X - (E|_{X\times 1}, H)_X = \deg E|_{X\times 0} - \deg E|_{X\times 1} = 0.$$

Finally, an arbitrary D can be expressed as a difference $H' - H''$ with H', H'' very ample. Indeed, let H be one fixed very ample divisor on X, which is possible since X is projective. Then in particular, H is ample, and so by definition of ampleness there is an $N \gg 1$ such that $D + NH$ is generated by global sections. Since H itself is very ample, it follows from [4, Ex. II.7.5(d)] that $H' = D + (N + 1)H$ and $H'' := (N + 1)H$ are both very ample as needed.

Given these notions, we define the subgroups $\operatorname{Pic}^0 X \subseteq \operatorname{Pic}^\tau X$ of $\operatorname{Pic} X$ to be the subgroups of algebraically and numerically trivially classes respectively.

Definition 1.1.2. The Néron-Severi group of X is defined to be $\operatorname{NS}(X) := \operatorname{Pic} X/ \operatorname{Pic}^0 X$, and the group of numerical classes on X is defined to be $\operatorname{Num}(X) := \operatorname{Pic} X/ \operatorname{Pic}^\tau X$.

In general, it is true that $\operatorname{NS}(X)$ is a finitely generated abelian group; however, the proof of this statement is very difficult. The rank

$$\rho(X) := \operatorname{rank} \operatorname{NS}(X)$$

is called the Picard rank of X. It is however, much easier to prove that when $\operatorname{char} k = 0$, the group of numerical equivalence classes $\operatorname{Num}(X)$ is free abelian of finite rank (see [4, Ex. V.1.8(b)]). Since we'll primarily be working with algebraic K3 surfaces over $k = \mathbb{C}$, this will suffice (see Proposition 2.1.3).

1.2 Preliminaries on Lattices

Definition 1.2.1. A lattice Λ is a free abelian group of finite rank endowed with a symmetric nondegenerate integral bilinear form

$$\langle , \rangle : \Lambda \times \Lambda \to \mathbb{Z}.$$

The lattice Λ is said to be even if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in \Lambda$; otherwise, it is said to be odd. The dual lattice

$$\Lambda^\vee := \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$$

is easily seen to be isomorphic to the lattice given by the following procedure. Consider the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Q}} \mathbb{Z}$. Since Λ is torsion-free, it follows that we have a natural embedding $\Lambda \hookrightarrow \Lambda_{\mathbb{Q}}$. We may then extend $\langle , \rangle$ to a symmetric nondegenerate rational bilinear form on $\Lambda_{\mathbb{Q}}$ and then consider the other manifestation of the dual lattice

$$\{x \in \Lambda_{\mathbb{Q}} : \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in \Lambda\}.$$

This identification is achieved by extending any $\varphi : \Lambda \to \mathbb{Z}$ to $\varphi_{\mathbb{Q}} : \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ and then using that we have a symmetric nondegenerate bilinear form on the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}}$ given by $\langle , \rangle$. With this identification, we immediately get a natural inclusion map $\Lambda \hookrightarrow \Lambda^\vee$. The discriminant group of Λ is defined to be the quotient group

$$\operatorname{disc}(\Lambda) := \Lambda^\vee/\Lambda.$$

Since both Λ and $\Lambda^\vee$ are free abelian groups of the same rank, it follows that $\operatorname{disc}(\Lambda)$ is a finite group, whose size is easily seen to be the absolute value of the determinant of any Gram matrix of Λ. Indeed, if we take a basis $\{\mathrm{e}_i\}$ of Λ and consider the corresponding Gram matrix $G = [\langle \mathrm{e}_i, \mathrm{e}_j\rangle]$, then if we identify Λ with $\mathbb{Z}^r$ ($r = \operatorname{rank}\Lambda$) via e_i, then $\Lambda^\vee \cong G^{-1}\mathbb{Z}^r$ and so $\operatorname{disc}(\Lambda) \cong \mathbb{Z}^r/G\mathbb{Z}^r$, and the result follows from elementary divisor theory of finite abelian groups. We say the lattice Λ is unimodular if $\operatorname{disc}(\Lambda) = \{0\}$, or equivalently if $\langle , \rangle$ is a unimodular bilinear pairing, i.e. that under some (and hence any) basis its Gram matrix has determinant ± 1. We define the length of the lattice Λ to be the length of its discriminant group as a finite $\mathbb{Z}$-module, i.e.

$$\ell(\Lambda) := \ell(\Lambda^\vee/\Lambda).$$

Finally, we call a lattice positive definite, negative definite, or indefinite according to the classification of the extension of $\langle , \rangle$ to $\Lambda_{\mathbb{R}}$ and talk of its signature as usual.

We'll need the following easy result:

Lemma 1.2.2. Let $\Lambda \hookrightarrow \Xi$ be an embedding of lattices (i.e. an embedding of abelian groups that preserves the bilinear form). Let $\Lambda^\perp := \{x \in \Xi : \langle x, y\rangle = 0 \text{ for all } y \in \Lambda\}$ be the orthogonal complement of Λ in Ξ. Then the following are equivalent:

(a) The lattice Λ satisfies $\Lambda = (\Lambda^\perp)^\perp$.
(b) There is some lattice $\Pi \subseteq \Xi$ such that $\Lambda = \Pi^\perp$.
(c) The image of Λ in Ξ is saturated, i.e. the cokernel Ξ/Λ is torsion free.

Proof. The implication (a) $\Rightarrow$ (b) is clear. The implication (b) $\Rightarrow$ (c) follows from the fact that if $x \in \Lambda$ is such that for some integer $n \geq 1$ we have $\langle nx, y\rangle = 0$ for all $y \in \Pi$, then already $\langle x, y\rangle = 0$ for all $y \in \Pi$. The implication (c) $\Rightarrow$ (a) follows from first consider the $\mathbb{Q}$-tensored embedding $\Lambda_{\mathbb{Q}} \hookrightarrow \Xi_{\mathbb{Q}}$ and then noting that under the natural embeddings of these lattices into their respective tensor products with $\mathbb{Q}$, we have $(\Lambda^\perp)^\perp_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}$ by linear algebra, from which it follows that if $x \in (\Lambda^\perp)^\perp$, then there is an $n \geq 1$ such that $nx \in \Lambda$. Since Ξ/Λ is torsion free, this implies $x \in \Lambda$ already, proving $(\Lambda^\perp)^\perp \subseteq \Lambda$, with the other inclusion being obvious. ∎

Definition 1.2.3. An embedding of lattices $\Lambda \hookrightarrow \Xi$ that satisfies the equivalent conditions of the previous lemma is said to be a primitive embedding.

In general for an embedding Λ, the double perp $(\Lambda^\perp)^\perp$ is called the primitive closure or saturation of Λ in Ξ, and is the smallest primitively embedded lattice containing it.

Finally, we will quote three useful theorems from the theory of lattices without proof. To state the first, let U be the hyperbolic plane which is $\mathbb{Z}^{\oplus 2}$ equipped with the intersection form that has Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

On the other hand, let $E_8(-1)$ be the usual E_8 matrix twisted by -1, i.e. the rank 8 lattice with intersection form

$$\begin{bmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

Then we have:

Theorem 1.2.4. Let Λ be an even indefinite unimodular lattice of signature (r, s) with $s \geq r$. Then $r \equiv s \pmod 8$ and Λ is isometric to

$$U^{\oplus r} \oplus E_8(-1)^{\oplus (s-r)/8}.$$

Proof. See [5, V.2.2]. The proof of this given there amounts to proving a uniqueness theorem of lattices of this type, and then noting that the lattice on the right is of this type. ∎

An example of this is:

Definition 1.2.5. The K3 lattice is defined to be

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8(-1)^2.$$

This is, up to isometry, the unique even indefinite unimodular lattice of signature $(3, 19)$. The relevance of this lattice will become clear below. The last theorem that we need is:

Theorem 1.2.6. Let Λ be an even lattice of signature (r, s). If $r \leq 3$, $s \leq 19$, and $\ell(\Lambda) \leq 22 - r$, then Λ admits a primitve embedding $\Lambda \hookrightarrow \Lambda_{\mathrm{K3}}$. Further, if in fact $r < 3, s < 19$ and $\ell(\Lambda) \leq 20 - r$, then this primitive embedding is unique up to isometries of Λ_{K3}.

Proof. See [6]. The first statement is contained in Cor. 1.12.3, and the second in Prop. 1.14.4. ∎

The relevance of these will become clear in Examples 2.4.9 and 2.4.10.

2 Introduction to K3 Surfaces

2.1 Algebraic K3 Surfaces

Definition 2.1.1. An algebraic K3 surface is a smooth projective surface X such that the canonical sheaf is trivial ($\omega_X \cong \mathcal{O}_X$) and the first cohomology group $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

If X is an algebraic K3 surface, it follows that $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. From Serre Duality, it follows that $h^2(X, \mathcal{O}_X) = h^0(X, \omega_X) = h^0(X, \mathcal{O}_X) = 1$, where in the last step we have used $\omega_X \cong \mathcal{O}_X$. It follows that the Euler characteristic

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 2.$$

When X is a K3, we have $\chi(X, \mathcal{O}_X) = 2$ and $K_X = 0$, so the classical Riemann-Roch theorem for X, [4, V.1.6], reduces to:

Proposition 2.1.2. Let X be an algebraic K3 surface. Then for any line bundle L on X we have

$$L^2 = 2\chi(X, L) - 4.$$

This has several interesting consequences in and of itself; e.g., this proves that an algebraic K3 X cannot admit a map of odd degree to projective space (and in particular cannot be embedded as an odd-degree subvariety of projective space), because otherwise the hyperplane class would give rise to a line bundle with odd self-intersection. From this proposition, another standard result follows:

Proposition 2.1.3. Let X be an algebraic K3 surface. Then $\mathrm{Pic}^\tau(X) = 0$. Equivalently, the natural surjections $\mathrm{Pic}\, X \to \mathrm{NS}(X) \to \mathrm{Num}(X)$ are isomorphisms.

Proof. Let $0 \neq L \in \mathrm{Pic}^\tau(X)$, so that in particular $L^2 = 0$. Since X is projective, X admits an ample line bundle H. Since L is numerically trivial, $L \cdot H = 0$. But this proves that $h^0(X, L) = 0$, since otherwise $L \cong \mathcal{L}(C)$ for some curve $C \subset X$, from which we'd get $L \cdot H > 0$, since an ample line bundle pairs with an effective divisor positively. The same applies to $-L$, so from Serre duality we conclude also that $h^2(X, L) = h^0(X, K_X - L) = 0$. It follows then that

$$L^2 = -2h^1(X, L) - 4 < 0,$$

which is a contradiction. ■

We end this section with an example.

Example 2.1.4. (Smooth Complete Intersection K3s) Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces $H_1, \dots, H_r \subset \mathbb{P}^n$ of degrees $d_1, \dots, d_r$ respectively with $d_1 \leq \dots \leq d_r$. Then we call X a smooth complete intersection of type $(d_1, \dots, d_r; n)$; note that $\dim X = n - r$ and $\deg X = d_1 \cdots d_r$. Further, X is nondegenerate iff $d_1 \geq 2$. We will show that the nondegenerate smooth complete intersection K3s surfaces are surfaces of type $(4; 3)$, $(2, 3; 4)$ and $(2, 2, 2; 5)$.

Suppose that X is a smooth complete intersection K3 of type $(d_1, \dots, d_r; n)$ with $0 \leq r \leq n$. Since X is a surface, $r = n - 2$. Next, a repeated application of the adjunction formula and the equality $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ gives that

$$\omega_X \cong \mathcal{O}_X\left(-n - 1 + \sum_{i=1}^r d_i\right),$$

so, since ω_X is trivial, we get $\sum_{i=1}^r d_i = n + 1$. Since each $d_i \geq 2$, it follows that

$$n + 1 = \sum_{i=1}^r d_i \geq 2r = 2n - 4,$$

which implies that $n \leq 5$. On the other hand, $\mathbb{P}^2$ itself has nonzero canonical sheaf, so $r \geq 1$ and $n \geq 3$. It follows immediately that the only possible values of $(d_1, \dots, d_r; n)$ for which X could be a K3 are $(4; 3)$, $(2, 3; 4)$, and $(2, 2, 2; 5)$, and for each of types X is a smooth surface with trivial canonical sheaf. It remains to show that in each of these cases, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, and this is proved in the following lemma.

Lemma 2.1.5. Let X be a smooth complete intersection of type $(d_1, \dots, d_r; n)$, where $n \geq 2$ and $0 \leq r \leq n - 1$, with $d_1 \leq \dots \leq d_r$. Then for each $\ell \in \mathbb{Z}$ and $1 \leq q \leq n - r - 1$, we have that $\mathrm{H}^q(X, \mathcal{O}_X(\ell)) = 0$. In particular, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ whenever $\dim X \geq 2$.

Proof. We induct on r, with the case $r = 0$ following from our knowledge of the cohomology $\mathrm{H}^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell))$ of projective spaces. Suppose that the statement has been proven for some n and r with $0 \leq r \leq n - 2$, and let X be a smooth complete intersection of type $(d_1, \dots, d_{r+1}; n)$. Then $X = Y \cap H$, where Y is a smooth complete intersection of type $(d_1, \dots, d_r; n)$ and H is a hypersurface of degree d_{r+1}. For each $\ell \in \mathbb{Z}$, we have the short exact sequence of sheaves

$$0 \to \mathcal{O}_Y(-d_{r+1} + \ell) \to \mathcal{O}_Y(\ell) \to \mathcal{O}_X(\ell) \to 0.$$

Then for any $1 \leq q \leq n - r - 2$, the long exact sequence in cohomology contains the terms

$$\mathrm{H}^q(Y, \mathcal{O}_Y(\ell)) \to \mathrm{H}^q(X, \mathcal{O}_X(\ell)) \to \mathrm{H}^{q+1}(Y, \mathcal{O}_Y(-d_{r+1} + \ell)).$$

By the inductive hypothesis, the outer two terms are zero. Therefore, it follows that so is the middle term. ■

2.2 Curves on K3 Surfaces

Suppose that C is a reduced connected projective curve. Then we recall that is arithmetic genus $p_a(C) := h^1(C, \mathcal{O}_C)$, and its geometric genus $p_g(C)$ is the genus of its normalization. If $\nu : \tilde{C} \to C$ denotes the normalization, then the cokernel K of $\mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}}$ is a skyscraper sheaf supported over the singular locus of C, and we get the short exact sequence

$$0 \to \bigoplus_{x \in C_{\mathrm{sing}}} \mathrm{H}^0(C, K_x) = \mathrm{H}^0(C, K) \to \mathrm{H}^1(C, \mathcal{O}_C) \to \mathrm{H}^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \to \mathrm{H}^1(C, K) = 0.$$

It follows that the arithmetic and geometric genera are related by

$$p_a(C) = p_g(C) + \sum_{x \in C_{\mathrm{sing}}} \delta_x$$

where the δ-invariant $\delta_x := h^0(C, K_x)$.

Now suppose that C lies on a smooth projective surface X. Then the adjunction formula tells us that

$$2p_a(C) - 2 = C \cdot (C + K_X).$$

When X is a K3, it follows that we have

$$C^2 = 2p_a(C) - 2 = 2p_g(C) - 2 + 2 \sum_{x \in C_{\mathrm{sing}}} \delta_x.$$

This already shows that given any (reduced, connected) curve C on a K3 surface, we must have $C^2 \geq -2$, and that the inequality is strict whenever C is singular. Since C^2 is always even, we have shown:

Proposition 2.2.1. Let X be a K3 surface. Then:

(a) X does not contain any (-1)-curves, and hence is minimal with respect to blow-ups in its birational class.
(b) If C is a reduced connected curve on X such that $C^2 = -2$, then C is a smooth rational curve.

2.3 Complex K3 Surfaces

Definition 2.3.1. A complex K3 surface is a compact connected 2-dimensional complex manifold X such that the canonical sheaf is trivial ($\omega_X \cong \mathcal{O}_X$) and the first cohomology group $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Serre's GAGA principle implies that if X is an algebraic K3 surface over $k = \mathbb{C}$, then the corresponding analytic space X^{an} is a complex K3 surface. Next, we collect some fundamental results about complex K3 surfaces.

Theorem 2.3.2. Let X be a complex K3 surface. Then the singular cohomology

$$\mathrm{H}^q(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } q = 0, 4, \\ \mathbb{Z}^{\oplus 22} & \text{if } q = 2, \text{ and} \\ 0, & \text{else.} \end{cases}$$

Proof. Firstly, it follows as in the previous section that $\chi(X, \mathcal{O}_X) = 2$. Since $\det \mathcal{T}_X \cong \omega_X^\vee \cong \mathcal{O}_X$, it follows that $c_1(X) := c_1(\mathcal{T}_X) = c_1(\det \mathcal{T}_X) = 0$, so that from Noether's Formula [7, I.5.5] it follows that the topological Euler characteristic $e(X) = c_2(X) = 24$. Next, we have that $\mathrm{H}^0(X, \mathbb{Z}) \cong \mathrm{H}^4(X, \mathbb{Z}) \cong \mathbb{Z}$, since X is a connected oriented manifold. Next, look at the long exact cohomology sequence corresponding to the exponential exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0.$$

Firstly $\mathrm{H}^0(X, \mathcal{O}_X) \twoheadrightarrow \mathrm{H}^0(X, \mathcal{O}_X^\times)$ given by $\mathbb{C} \twoheadrightarrow \mathbb{C}^\times$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ tells us that $\mathrm{H}^1(X, \mathbb{Z}) = 0$. By Poincaré duality, it follows that

$$0 = \operatorname{rank} \mathrm{H}^1(X, \mathbb{Z}) = \operatorname{rank} \mathrm{H}_1(X, \mathbb{Z}) = \operatorname{rank} \mathrm{H}^3(X, \mathbb{Z})$$

so that $\mathrm{H}^3(X,\mathbb{Z})$ is torsion and so again by Poincaré duality we have

$$\mathrm{H}^3(X,\mathbb{Z}) \cong \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}} \cong \mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}.$$

However, we have:

Lemma 2.3.3. If X is a complex K3 surface, then $\mathrm{H}_1(X,\mathbb{Z})$ is torsion-free.

Proof. Otherwise, there would be some integer $n \geq 2$ and a surjection $\mathrm{H}_1(X,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}/n$, which would give a surjection $\pi_1(X) \twoheadrightarrow \mathbb{Z}/n$. The kernel of this map would correspond to a degree n topological covering space $\pi : Y \to X$, which would then satisfy $e(Y) = ne(X) = 24n$. On the other hand, since π is a topological covering map, we have $\omega_Y \cong \pi^*\omega_X$ so that $\omega_Y \cong \mathcal{O}_Y$ and hence $h^2(Y,\mathcal{O}_Y) = h^0(Y,\omega_Y) = 1$. From Noether's formula, we'd get as before that $2 - h^1(Y,\mathcal{O}_Y) = \chi(Y,\mathcal{O}_Y) = (1/12)c_2(Y) = 2n \Rightarrow h^1(Y,\mathcal{O}_Y) = 2 - 2n$. But this is a contradiction, since the left hand side is nonnegative and the right hand side negative. ■

It follows from this that $\mathrm{H}^3(X,\mathbb{Z}) = \mathrm{H}_1(X,\mathbb{Z}) = 0$. Then from the Universal Coefficient Exact sequence

$$0 \to \mathrm{Ext}^1(\mathrm{H}_1(X,\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_2(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

it follows that $\mathrm{H}^2(X,\mathbb{Z})$ is torsion free and hence free abelian. Since the topological Euler characteristic $e(X) = 24$, it follows that $\operatorname{rank}\mathrm{H}^2(X,\mathbb{Z}) = e(X) - \operatorname{rank}\mathrm{H}^0(X,\mathbb{Z}) - \operatorname{rank}\mathrm{H}^4(X,\mathbb{Z}) = 22$.

■

2.4 Lattice and Hodge Theory for K3 Surfaces

Recall that Poincaré duality equips $\mathrm{H}^2(X,\mathbb{Z})$ with a unimodular bilinear paring $B : \mathrm{H}^2(X,\mathbb{Z})\times\mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{Z}$. Since X is in particular a closed real 4-manifold, we can apply the Hirzebruch Signature Formula to give us the signature of this pairing

$$\sigma(X) = \langle L_1(p_1(X)), [X]\rangle = \left\langle \frac{1}{3}p_1(X), [X] \right\rangle = \frac{1}{3}(c_1(X)^2 - c_2(X)) = -16.$$

This along with $\operatorname{rank}\mathrm{H}^2(X,\mathbb{Z}) = 22$ tells us that this bilinear paring B has signature $(3,19)$. Finally:

Lemma 2.4.1. The bilinear pairing B is even, i.e. for any $x \in \mathrm{H}^2(X,\mathbb{Z})$ we have $B(x,x) \equiv 0 \pmod 2$.

Proof. Let $w, v \in \mathrm{H}^*(X,\mathbb{Z}/2)$ and be the total Stiefel-Whitney and Wu classes (of the tangent bundle to X) respectively; these are related by $w = \mathrm{Sq}(v)$, where Sq denotes the total Steenrod square operation. In particular, we have $v_1 = w_1 = 0$ (since X is a complex manifold) and $w_2 = v_2 + v_1^2 = v_2$. Next, the defining property of the Wu class tells us that if $y \in \mathrm{H}^2(X,\mathbb{Z}/2)$ is any other element, then

$$w_2 \smile y = v_2 \smile y = \mathrm{Sq}^2\, y = y^2.$$

Since the first Chern class (also of the tangent bundle) $c_1 = 0$, as established above, it follows from $w_2 \equiv c_1 = 0 \pmod 2$ that for any element $y \in \mathrm{H}^2(X,\mathbb{Z}/2)$ we have $y^2 = 0$. But now the cohomology of X is free abelian in all ranks, and so the Universal Coefficients map $\mathrm{H}^*(X,\mathbb{Z}) \otimes \mathbb{Z}/2 \to \mathrm{H}^*(X,\mathbb{Z}/2)$ is an isomorphism. In particular, if $x \in \mathrm{H}^2(Z,\mathbb{Z})$ is any element and $y = \overline{x}$ is its mod 2 reduction, then $y^2 = 0$ says exactly that $B(x,x) = x \smile x \in \mathrm{H}^4(X,\mathbb{Z})$ is zero mod 2, as needed. ■

This is a complex analytic analog of Proposition 2.1.2. It follows that $\mathrm{H}^2(X,\mathbb{Z})$ is an even indefinite unimodular lattice of signature $(3,19)$. It follows from Theorem 1.2.4 quoted above that:

Theorem 2.4.2. Let X be a complex K3. Then the lattice $\mathrm{H}^2(X,\mathbb{Z})$ is an even indefinite unimodular lattice of signature $(3,19)$ and hence isometric to the K3 lattice

$$\Lambda_{\mathrm{K3}} \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

From now on, let X be an complex algebraic K3 surface, so that X is projective and hence Kähler, and hence $\mathrm{H}^*(X, \mathbb{C})$ admits a Hodge structure.[1] We have the following theorem:

Theorem 2.4.3. Let X be an complex algebraic K3. Then the Hodge diamond of X is given by the following:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof. Most of the Hodge diamond of X follows immediate from Theorem 1.3.2; the only thing left to know is $h^{0,2}$, which we compute using

$$h^{0,2} = h^0(X, \Omega_X^2) = h^0(X, \omega_X) = h^0(X, \mathcal{O}_X) = 1.$$

From $b_2(X) = 22$, it follows then that $h^{1,1} = 20$. ■

We know that $\mathrm{H}^2(X, \mathbb{Z})$ admits a a Hodge structure of weight 2, and we can write

$$\mathrm{H}^2(X, \mathbb{C}) \cong \mathrm{H}^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{0,2}(X),$$

where the outer two pieces are one-dimensional and complex conjugates. The cup product on $\mathrm{H}^2(X, \mathbb{Z})$ extends to a bilinear pairing on $\mathrm{H}^2(X, \mathbb{C})$ given by $\langle \alpha, \beta \rangle \mapsto \alpha \cdot \beta = \int_X \alpha \wedge \beta$. If we write $\mathrm{H}^{2,0}(X) = \mathbb{C}\sigma_X$ for some choice of σ, then $\mathrm{H}^{0,2}(X) = \mathbb{C}\overline{\sigma}_X$ and the Hodge-Riemann bilinear relations say that:

(a) $\sigma_X^2 = 0$,
(b) $\sigma_X \cdot \overline{\sigma}_X > 0$, and
(c) $\mathrm{H}^{1,1}(X)^\perp = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X)$.

The first and third of these follow from considering bidegrees, and (b) follows from the fact that $\overline{\sigma}_X$ agrees with the Hodge star $*\sigma_X$ of σ, at least upto a positive real constant. This set-up allows us to study the moduli theory of K3 surfaces in detail, as follows.

Definition 2.4.4. A marked K3 surface is a pair (X, Φ) where X is a complex K3 surface and Φ is a chosen isomorphism

$$\Phi : \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}.$$

An isomorphism of marked K3 surfaces is defined to be an isomorphism of the underlying K3 surfaces that preserves the marking. Next, note that we can produce a map, called the period map,

$$\{\text{isomorphism class of marked K3 surfaces}(X, \Phi)\} \to \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$$

given by sending (X, Φ) to the complex line $\Phi(\mathrm{H}^{1,1}(X)) \subseteq \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$. By the Hodge-Riemann bilinear relations, the image of this map is contained in the subset

$$\mathbb{D}(\Lambda_{\mathrm{K3}}) := \{\sigma \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) : \sigma^2 = 0, \sigma \cdot \overline{\sigma} > 0\}.$$

This is called the period domain for K3 surfaces. Since it is an open subset of a smooth quadric hypersurface in $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})^{21}$, it follows that $\mathbb{D}(\Lambda_{\mathrm{K3}})$ is a 20-dimensional smooth quasiprojective variety. The fundamental result of the moduli theory of K3 surfaces is:

[1] By a theorem of Siu (see [7, §IV.3]), any complex K3 X is Kähler, and so the cohomology $\mathrm{H}^*(X, \mathbb{C})$ admits a Hodge Decomposition. Siu's proof is way beyond the scope of this article.

Principle 2.4.5 (Torelli)**.** The map

$$\{\text{isomorphism class of marked K3 surfaces}(X, \Phi)\} \to \mathbb{D}(\Lambda_{\mathrm{K3}})$$

defined above is a bijection.[2]

The reasoning for introducing a marking is that it is in general easier to work with the moduli spaces of marked K3s (with various additiona specifications), and then to consider the moduli space of all K3s to mod out by an appropriate action by the orthogonal group $\mathrm{O}(\Lambda_{\mathrm{K3}})$.

Finally, note also that the long exact sequence in cohomology gives us an embedding

$$c_1 : \mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathcal{O}_X^\times) \hookrightarrow \mathrm{H}^2(X, \mathbb{Z}).$$

It is clear from Chern-Weil Theory that c_1 maps $\mathrm{Pic}(X)$ to the Hodge classes

$$\mathrm{Hdg}^1(X) := \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{Z}).$$

The Lefschetz (1,1) principle, which is the $(1,1)$ case of the Hodge conjecture over $\mathbb{Z}$ and whose proof we recall here below, says that c_1 is surjective onto this lattice; this is true for any arbitrary compact Kähler manifold X.

Proposition 2.4.6 (Lefschetz (1,1) Principle)**.** Let X be any compact Kähler manifold. The Chern class map $c_1 : \mathrm{Pic}(X) \to \mathrm{Hdg}^1(X)$ is surjective.

Proof. Consider once again the exponential long exact sequence in cohomology, which contains the terms

$$\mathrm{H}^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{i_*} \mathrm{H}^2(X, \mathcal{O}_X).$$

Since X is compact Kähler, we have an identification $\mathrm{H}^2(X, \mathcal{O}_X) \cong \mathrm{H}^{0,2}(X)$, and under this identification i_* is given by the restriction of the projection map $\mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^{0,2}(X)$, so $\mathrm{Hdg}^1(X) \subseteq \ker i_* = \operatorname{im} c_1$. ∎

In general, the image of the map c_1 is called the Néron-Severi Lattice $\mathrm{NS}(X)$ of X. When X is a complex algebraic K3 surface, this coincides with the previous definition by Proposition 2.1.3 and Serre's GAGA principle. The above argument shows that in the case of a complex algebraic K3 X, the first Chern class gives an isomorphism $c_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X) = \mathrm{Hdg}^1(X)$. Recall that the Picard rank of X is

$$\rho(X) = \operatorname{rank} \mathrm{NS}(X).$$

It follows immediately from the above discussion that

$$\rho(X) = \dim_{\mathbb{C}} \mathrm{NS}(X)_{\mathbb{C}} \leq \dim_{\mathbb{C}} \mathrm{H}^{1,1}(X, \mathbb{C}) =: h^{1,1}(X) = 20,$$

which shows that the Picard rank of a complex algebraic K3 (and hence by the Lefschetz principle, that of any algebraic K3 over any field of characteristic zero) is at most 20.[3] Again from Proposition 2.1.3 and the Hodge Index Theorem [4, V.1.9], it follows that the lattice $\mathrm{NS}(X)$ has index $(1, \rho(X) - 1)$. Finally, since $\mathrm{NS}(X)$ is given by the intersection with a subspace of the complexification, it follows immediately that $\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X, \mathbb{Z})$ is a primitive embedding.

The above discussion gives rise to the natural question: which sublattices $\Lambda \hookrightarrow \Lambda_{\mathrm{K3}}$ occur as Néron-Severi lattices of K3 surfaces? The above discussion certainly gives necessarily conditions; namely, Λ needs to be primitive and of signature $(1, \rho - 1)$, where $\rho = \operatorname{rank} \Lambda$. Further, Λ must contain some class that can be an ample class; equivalently, if there is some $H \in \Lambda$ such that $H^2 > 0$. It turns out at these conditions are also sufficient, at least for a somewhat weaker result. To state this result, more precisely, we introduce the notion of a polarization.

[2]This is not strictly true, since one must account for automorphisms; this is why we call it is more of a guiding "principle" rather than a theorem. A precise formulation can be found in [3, Prop. 7.2.1]. Another precise formulation in a slightly different direction is given below, see Theorem 2.4.8.

[3]It is interesting to contrast this with the situation in positive characteristic, where it is again true that $\rho(X) \leq 22$, which can be proven using ℓ-adic étale cohomology just as it is for complex cohomology above, but where the bound $\rho(X) = 22$ can in fact be achieved. Algebraic K3 surfaces that achieve this bound are said to be *supersingular*, and supersingular K3 surfaces admit a rich theory [8].

Definition 2.4.7. Given any primitive sublattice $\Lambda \hookrightarrow \Lambda_{\mathrm{K3}}$, a Λ-polarized (marked) K3 is a marked K3 (X, Φ) such that under the isomorphism $\Phi : \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}$, we have $\Lambda \hookrightarrow \Phi(\mathrm{NS}(X))$, and further that under this inclusion, Λ contains an ample class of X. Given an arbitrary Λ, a K3 surface X is said to admit a Λ-polarization if for some choice of marking, i.e. isomorphism, $\Phi : \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}$, the pair (X, Φ) is a Λ-polarized marked K3.

Then we have the following difficult but quite amazing theorem:

Theorem 2.4.8 (Moduli Theory of K3 Surfaces)**.** Let Λ be an even integral lattice of signature $(1, s)$ for some $s \geq 0$. If Λ admits a primitive embedding into Λ_{K3}, then given a fixed primitive embedding $\Lambda \hookrightarrow \Lambda_{\mathrm{K3}}$, the Λ-polarized K3 surfaces admit a local moduli space of dimension $19 - s$.

Proof. See [9, Thm. 10.1]. ■

This moduli space depends on choice of primitive embedding $\Lambda \hookrightarrow \Lambda_{\mathrm{K3}}$ up to isometry, so when Λ admits a unique primitive embedding into Λ_{K3} up to isometry (such as in cases covered by Theorem 1.2.6), there is a unique moduli space of Λ-polarized K3 surfaces (which we can then speak of without explicitly mentioning the primitive embedding).

Example 2.4.9. In general, a polarized K3 surface of degree $2d$ (with $d \geq 1$ an integer) is a pair (X, L) where X is a K3 and L is a primitive ample line bundle with $L^2 = 2d$. For any $d \geq 1$ the rank-one lattice Ξ_{2d} with Gram matrix $[2d]$ is an even lattice of signature $(1, 0)$ and length 1, and hence by Theorem 1.2.6 admits a unique primitive embedding into Λ_{K3} upto isometry. Clearly, a marked polarized K3 surface of degree $2d$ is the same as a Ξ_{2d}-polarized K3 surface. The above theorem states that marked polarized K3 surfaces of degree $2d$ admit a local moduli space of dimension $19 - 0 = 19$. In fact, it is possible to see directly what such a moduli space should be. Indeed, let Λ_{2d} denote the orthogonal complement of Ξ_{2d} in Λ_{K3}. Then it follows again from the Hodge-Riemann bilinear relations, that the period map restricted to isomorphism classes of marked polarized K3 surface of degree $2d$ has image contained in the hyperplane section $\mathbb{D}(\Lambda_{2d}) := \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) \cap \mathbb{D}(\Lambda_{\mathrm{K3}})$ of the usual period domain. One can check that the resulting scheme $\mathbb{D}(\Lambda_{2d})$ is still a smooth quasiprojective variety, and consequently of dimension 19. This is the local moduli space referred to in the previous theorem.

Example 2.4.10. Now consider the hyperbolic plane U. This is an even unimodular lattice of signature $(1, 1)$ and length, and hence admits a unique primitive embedding into Λ_{K3} up to isometries by Theorem 1.2.6. Then Theorem 2.4.8 tells us that there is a unique local moduli space of U-polarized K3 surfaces of dimension 18. We will see below that this is nothing but the space of (marked) elliptic K3 surfaces (Theorem 3.2.4).

3 Elliptic K3 Surfaces

In this section we recall the basic theory of elliptic surfaces as in [1] and then prove our main result, namely Theorem 3.2.4.

3.1 Introduction to Elliptic Surfaces

Recall that k is an algebraically closed base field.

Definition 3.1.1. Let C be smooth projective curve. A elliptic surface over C is a smooth projective surface S with an elliptic fibration over C, i.e. a smooth surjective morphism

$$f : S \to C$$

with connected fibers such that

(a) all but finitely many of the fibers of f are smooth curves of genus 1, and
(b) no fibre contains an exceptional curve of the first kind (i.e. a smooth rational curve of self intersection -1).

Further, by convention we require

(c) that the fibration admits a section, i.e. there is a map $\sigma : C \to S$ with $f \circ \sigma = 1_C$, and
(d) the fibration has at least one singular fiber.

The condition (a) defines an elliptic fibration. Note that (-1)-curves arise naturally as exceptional divisors of blow-ups of surfaces at smooth point, so this condition (b) is a relative minimality condition (which does not imply an absolute minimality condition if one forgets the fibration structure). The condition (c) allows us to think of (almost all of) the fibers of f as not just genus 1 curves but rather as elliptic curves, with the distinguished rational point for the fiber S_t for a $t \in C(k)$ being $\sigma(t)$. Finally, the condition (d) is needed to exclude trivial fibrations $S = C \times E \xrightarrow{\pi_1} C$ with E an elliptic curve, for which our finiteness results don't usually hold.

Note that in this case, if $K = k(C)$ is the function field of C, then the generic fiber of f, i.e. the fiber S_η of f over the generic point $\eta \in C$ of C, is also a genus 1 curve E over K equipped with the K-rational point σ, and is hence an elliptic curve E/K. In general, sections of f correspond exactly to K-rational points of E:

Proposition 3.1.2. The global sections of f are in bijective correspondance with $E(K)$.

Proof Sketch. Given any section σ, define the point of $E(K)$ by taking the image $\sigma(C)$ and intersecting it with $S_\eta = E$. Conversely, given a K-rational point P, consider its Zariski closure in S. This is a curve Γ, since its function field is K, which has transcendence degree 1 over k. Restricting the fibration $f|_\Gamma : \Gamma \to C$ we get a finite, birational morphism onto the nonsingular curve C and so by Zariski's main theorem for curves, this $f|_\Gamma$ is an isomorphism. The inverse of $f|_\Gamma$ is then the required section. The correspondence is easily seen to be bijective. For details, we refer to [1, Prop. 5.4]. ■

We can also go in the other direction:

Definition 3.1.3. Let E be an elliptic curve over the function field K of a curve C. If there is an elliptic surface $f : S \to C$ whose generic fiber is isomorphic to E/K, then we say that f is a Kodaira-Néron Model for E/K, call S the elliptic surface associated with E/K.

The Kodaira-Néron model is characterized by the following universal property: if S' is another projective surface and if $f' : S' \to C$ has generic fiber E/K, then f' factors through f, i.e. there is a morphism $g : S' \to S$ such that $f' = f \circ g$. This is a statement of the relative minimality of Kodaira-Néron models.

Proposition 3.1.4. Given an elliptic curve E/K, the Kodaira-Neron model exists and is unique up to isomorphisms.

Proof Sketch. Take a generalized Weierstraß equation for E/K in $\mathbb{P}^2_K$ that looks like

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let $\Sigma \subset C$ be a finite subset containing all the poles of the coefficients $a_i \in K = k(C)$ and the zeroes of the discriminant Δ of E/K. Let $C^\circ := C \setminus \Sigma$, and let $S^\circ \subset \mathbb{P}^2 \times C^\circ$ be the open surface defined by the above equation; the projection to C° is a smooth elliptic fibration. Let $\tilde{S} \subset \mathbb{P}^2 \times C$ be the Zariski closure of S°; this is a possibley singular surface, but its fibers over $t \in \Sigma$ are either nodal or cuspidal curves, being degenerations of elliptic curves. The singular points of $\tilde{S}$ are contained in the set of nodes and cusps of nonelliptic fibers, and hence are finite in number. By resolving these, we get a smooth projective surface $\overline{S}$ and a birational morphism $\overline{S} \to \tilde{S}$. The composition $\overline{S} \to C$ may not be relatively minimal, but by Castelnuovo's Theorem, we can keep contracting the finitely many (-1)-curves to finally arrive at an elliptic fibration $S \to C$ with generic fiber E/K. The uniqueness of the Kodaira-Néron model follows from its universal property. Again, for full details we refer to [1, Thm. 5.19]. ■

This tells us that studying elliptic surfaces is equivalent to studying elliptic curves over function fields of curves. We end this section with detailed example:

Example 3.1.5. Consider a pencil $V^2 \subset \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ of plane cubics whose base locus consists of 9 distinct points, so that V contains smooth cubic curves. If $f, g \in V$ are any two linearly independent elements, then the surface $S \subset \mathbb{P}^2 \times \mathbb{P}^1$ defined by

$$S = \mathbb{V}(sf + tg) \subset \mathbb{P}^2 \times \mathbb{P}^1$$

where $[s,t]$ are coordinates on the $\mathbb{P}^1$ is the blow-up of $\mathbb{P}^2$ at the nine points of the cubic pencil spanned by f and g, and is in particular a smooth surface, whose projection onto the second factor gives an elliptic fibration. Any base point of the pencil gives a section of this fibration, and the fibration clearly has a singular fiber, since the discriminant of the curve $sf + tg$ is a nonzero polynomial in s and t. When we can show that no fibres contain (-1)-curves, this gives us an example of an elliptic surface over $\mathbb{P}^1$. We'll carry this out explicitly next.

Suppose now that $\operatorname{char} k \neq 3$, and take $f = x^3 + y^3 + z^3$ and $g = -xyz$, so that S is defined by the equation

$$s(x^3 + y^3 + z^3) - txyz = 0.$$

It is easy to check that this pencil is singular, and hence the corresponding fiber of S singular, exactly when $s = 0$ or $t^3 = 27s^3$, in both of which cases the fibers are "triangles", i.e. three nonconcurrent lines (this needs $\operatorname{char} k \neq 3$). We claim that if T denotes one of these lines, then $T^2 = -2$. From this it would follow that the corresponding $S \to \mathbb{P}^1$ is an elliptic fibration. Indeed, to show this, suppose that F is a smooth fiber and F' is the singular fiber with $F' = T + T' + T''$. Then from $F \sim F'$, we conclude that

$$0 = (F,T) = (F',T) = T^2 + (T,T') + (T,T'') = T^2 + 2$$

because $(T,T') = (T,T'') = 1$. The result follows. An explicit computation of the Kodaira-Néron Model of this surface can be found in [1, Example 5.24].

We end with computing the self-intersection of any section with respect to the invariant $\chi(S, \mathcal{O}_S)$. For this we need:

Theorem 3.1.6 (Canonical Bundle Formula)**.** The canonical bundle of an elliptic surface $f : S \to C$ is given by

$$\omega_S = f^*(\omega_C \otimes \mathscr{L}^{-1})$$

where $\mathscr{L}$ is some line bundle of degree $-\chi(S, \mathcal{O}_S)$ on C. In particular, we have

$$K_S \equiv (2g(C) - 2 + \chi(S, \mathcal{O}_S))F \text{ and } K_S^2 = 0.$$

Proof. See [1, Thm. 5.44] for a proof sketch and references to the complete proof. ■

In the theory of elliptic K3 surfaces, this can be circumvented by noting (Lemma 3.2.1) that ellpitic K3 surfaces must have base curve $\mathbb{P}^1$, and then working with minimal Weierstraß models to prove this directly (see [1, Ch. 5]). We will content ourselves with quoting it and noting the following consequence:

Corollary 3.1.7. Let $f : S \to C$ be an elliptic surface. Then given any section $\sigma : C \to S$, or equivalently $P \in E(K)$, if we denote by P also the corresponding divisor on S, then $P^2 = -\chi(S, \mathcal{O}_S)$.

Proof. First note that P is a smooth curve. Next, the adjunction formula gives us

$$2g(C) - 2 = 2g(P) - 2 = P^2 + P \cdot K_S = P^2 + 2g(C) - 2 + \chi(S, \mathcal{O}_S),$$

where in the last step we have used $P \cdot F = 1$. ■

3.2 Main Theorem

Given our discussion so far, a natural question to ask is: when does a K3 surface X admit a elliptic fibration? The first thing to say about this is:

Lemma 3.2.1. Suppose we have a surjective morphism $\pi : X \to C$ from a K3 surface X to a smooth projective curve C. Then $C \cong \mathbb{P}^1$.

Proof. The Leray Spectral Sequence for this map and the sheaf $\mathcal{O}_X$ on X has $E_2^{p,q} = \mathrm{H}^p(C, (\mathrm{R}^q\pi_*)\mathcal{O}_X)$ and converges to $E_\infty^{p+q} = \mathrm{H}^{p+q}(X, \mathcal{O}_X)$. Since C is a curve, this spectral sequence degenerates already at the second page, and we get an embedding $\mathrm{H}^1(C, \pi_*\mathcal{O}_X) \hookrightarrow \mathrm{H}^1(X, \mathcal{O}_X)$. Since X and C are smooth, it follows that $\pi_*\mathcal{O}_X = \mathcal{O}_C$. Since X is a K3, we have $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, and it follows that $\mathrm{H}^1(C, \mathcal{O}_C) = 0$ and hence $g(C) = h^1(C, \mathcal{O}_C) = 0$ as well. ∎

It follows from this that if $\pi : X \to C$ is an elliptic K3 surface, then for any fiber $P \in E(K)$, we have $P^2 = -\chi(X, \mathcal{O}_X) = -2$ (using Theorem [to cite] and $\chi(X, \mathcal{O}_X) = 2$ proven above). Therefore, a necessarily condition for a K3 surface to admit an elliptic surface structure is the existence of a (-2)-curve, which is then automatically smooth and rational by §2.2. Therefore, it makes sense to study (-2)-divisors on X, which we now do.

Lemma 3.2.2. Let X be a K3 surface and let D be a divisor. If $D^2 \geq -2$, then either D or $-D$ is effective.

Proof. By Proposition 2.1.2, we have $D^2 = 2\chi(X, D) - 4$, so that $D^2 \geq -2$ implies $\chi(X, D) > 0$, which implies that $h^0(X, D) + h^2(X, D) > 0$. Since $h^2(X, D) = h^0(X, K_X - D) = h^0(X, -D)$ by Serre duality, it follows that at least one of $h^0(X, D)$ or $h^0(X, -D)$ is positive. Finally, a divisor D is effective iff $h^0(X, D)$ is positive. ∎

Lemma 3.2.3. Let D be an effective divisor on a K3 surface X satisfying $D^2 = -2$. Then D is supported on (-2)-curves, i.e. there is an integer $r \geq 1$ and (-2)-curves $C_1, \ldots, C_r \subset X$ and integers $m_1, \ldots, m_r \in X$ such that $D = \sum_{i=1}^r m_i C_i$.

Proof. Fix a polarization (i.e. projective embedding) of X so we may talk of degrees of divisors. Write $D = \sum_C m_C C$, where the C are reduced and connected curves and $m_C \geq 0$ by effectiveness. Then $D^2 < 0$ implies there is some C_1 with $m_{C_1} > 0$ such that $D \cdot C_1 < 0$, and from this it follows that in fact $C_1^2 < 0$. Consequently, $C_1^2 = -2$. Let $D_1 := D + (D \cdot C_1)C_1$; this amounts to reflection in C_1. Then $D_1^2 = D^2$. Then D_1 is a nonzero divisor with $D_1^2 \geq -2$, so by the previous lemma, either D_1 or $-D_1$ is effective. If D_1 is effective, then $\deg(D_1) < \deg D$, and so we many induct on degree to deal with this case. The only case that remains to be done is when D_1 is antieffective. We claim that in this case we have $\deg(-D_1) \leq \deg D$. To show this, write

$$D = \sum_{C \neq C_1} m_C C + m_{C_1} C_1 \text{ and so } D_1 = \sum_{C \neq C_1} m_C C + (m_{C_1} + (D \cdot C_1))C_1.$$

Next, note that

$$D \cdot C_1 = \sum_{C \neq C_1} m_C (C \cdot C_1) - 2m_{C_1}$$

with each $(C \cdot C_1)$ for $C \neq C_1$ being positive. It follows that

$$\begin{aligned}
\deg(-D_1) &= -\sum_{C \neq C_1} m_C \deg(C) + (-(D \cdot C_1) - m_{C_1}) \deg(C_1) \\
&\leq 0 + (m_{C_1} - \sum_{C \neq C_1} m_C (C \cdot C_1)) \deg(C_1) \\
&\leq \sum_{C \neq C_1} m_C \deg(C) + m_{C_1} \deg(C_1) = \deg(D).
\end{aligned}$$

Next, notice that $-D_1$ and D have completely symmetric roles to conclude that also $\deg(D) \leq \deg(-D_1)$ and so equality must hold everywhere, and consequently $m_C = 0$ for all $C \neq C_1$, i.e. $D = m_{C_1} C_1$. Further, since $D^2 = -2$, it follows that $m_{C_1} = 1$ as well, so that in this case we simply get that $D = C_1$, completing the proof.

∎

Theorem 3.2.4. Let X be an complex algebraic K3 surface. Then the following are equivalent:

(a) X is an elliptic surface.
(b) X admits divisors D and F with $D^2 = 0$, $D \cdot F = 1$, and $F^2 = 0$.

(c) X admits a U-polarization.

Proof.

(a) $\Rightarrow$ (b) Let P be a section and F be a fiber. It suffices to take $D = P + F$.

(b) $\Leftrightarrow$ (c) This is immediate, since the sublattice spanned by D and F is exactly U, and primitivity of the embedding is a consequence of $D \cdot F = 1$.

(b) $\Rightarrow$ (a) We give a detailed sketch. By Lemma 3.2.2, we conclude that D or $-D$ is effective. By $D \cdot F = 1$, it follows that D is primitive in $\mathrm{NS}(X)$. Then linear system $|D|$ may still contain nonempty base locus, but this base locus consists exclusively of (-2)-curves by the same reasoning as in the previous lemma, and so they can be eliminated by successive reflections in (-2)-curves as above, with the degree dropping at each stage. Therefore, this procedure stops after finitely many steps. Let s be the procedure so applied, and let $D' := s(D)$. Then D' is isotropic and effective such that $|D'|$ is basepoint free. Then it remains to check that $h^0(X, D') = 2$, and the resulting map $\pi = \varphi_{|D'|} : X \to \mathbb{P}^1$ is a genus one fibration, i.e. satisfying condition (a) in the definition of an elliptic fibration. Since K3 surfaces are globally minimal (Proposition 2.2.1(a)), it follows that (b) is automatic. Since D' is primitive, it features as one of the singular fibres of this fibration; it follows that (d) is satisfied. It remains to show (c), i.e. that this fibration admits a section. Next, set $F^2 = 2r$ with $r \in \mathbb{Z}$, and define $F' =: s(F) - (r+1)D'$. Then $(F')^2 = -2$ and $F' \cdot D' = 1$. By Lemma 3.2.2, F' is either effective or anti-effective, so $F' \cdot D' = 1$ and D' effective implies that F' is also effective. By Lemma 3.2.3, $F' = \sum_{i=1}^{r} m_i C_i$ for some $m_i > 0$ and (-2)-curves C_i. Since D' is a fiber, it is in particular nef, so we deduce from $F' \cdot D' = 1$ that F' is simply a (-2)-curve C, which is a smooth rational curve. It follows that the fibration given by $|D'|$ admits the section C, satisfying (c).

■

The key point here is the X admits an genus one fibration iff there is a nonzero D such that $D^2 = 0$, and then it admits a section iff there is additionally an F meeting D once. There are also examples of K3 surfaces admitting genus one fibrations but not sections, and therefore not fitting into our definition; the Fermat quartic surface $\mathbb{V}(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}$ is one (see [1, Example 11.30]).

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