The Grassmannian and its Schubert Cell Decomposition

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Abstract

This paper is written in fulfillment of the requirements of a Freshman Seminar 51E: The Story of the Alternating Sign Matrix Conjecture offered at Harvard during the fall of 2019 by Prof. Lauren Williams. We introduce the Grassmannian $\mathrm{Gr}(k, n)$ and discuss its basic properties, including the Plücker embedding and Plücker coordinates, using machinery from algebraic geometry and multilinear algebra. Finally, we talk about Schubert cells and the decomposition of the Grassmannian into Schubert cells, concluding with a combinatorial perspective on the bijection between Schubert cells and Young diagrams that fit in a $k \times (n - k)$ rectangle.

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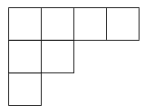
1 A Brief Word about Notation

In the following discussion, we adopt the following notation.

1. Let F be field of characteristic zero. In practice, we take $F = \mathbb{R}$ or $F = \mathbb{C}$.
2. For positive integer $n \in \mathbb{Z}^{>0}$, let $[n] := \{1, 2, \ldots, n\}$.
3. For set S and positive integer $k \in \mathbb{Z}^{>0}$, let $\binom{S}{k}$ denote the set of subsets of S of size K. Then $\left|\binom{S}{k}\right| = \binom{|S|}{k}$.
4. A Young diagram of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ means an array of left-aligned boxes with k rows such that for $i \in [k]$, there are λ_i boxes in the i^{th} row. For example, the Young diagram of shape $(4, 2, 1)$ looks like:



   For a young diagram Y of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define the size $|Y|$ of Y as the number of boxes in Y, i.e. $|Y| = \sum_{i=1}^{k} \lambda_i$.
5. A Young *tableu* of shape λ is a Young diagram of shape λ filled with entries from some specified set. The following is an example of a Young tableu of shape $(4, 2, 1)$:

1	2	3	4
2	3		
4			

2 Introduction to Projective Varieties

We begin our discussion by talking about projective spaces and varieties. The following has been principally taken from Harris[1].

Definition 2.1. Given a vector space V over the field F, the projective space $\mathbb{P}V$ is defined to be the set of 1-dimensional vector subspaces of V. If V has finite dimension $n+1$, then we write $\mathbb{P}V = \mathbb{P}^n_F$, or, if F is understood, simply $\mathbb{P}^n$.

Equivalently, we can think of $\mathbb{P}^n$ as the complement of the origin in F^{n+1}, modulo the equivalence relation $\sim$ induced by the action of the group $F^\times$ via scalar multiplication. Mathematically, $\mathbb{P}^n = F^{n+1} \setminus \{0\}/\sim$, where $x \sim y \iff \exists \lambda \in F^\times$ s.t. $x = \lambda y$. For nonzero $v \in V$, we let the line spanned by v denoted by $[v]$ as an element of $\mathbb{P}V$. Likewise, an element $\mathbb{P}^n$ is written in homogenous coordinates $[z_0, z_1, \ldots, z_n]$, which means the line spanned by $(z_0, z_1, \ldots, z_n) \in F^{n+1}$.

The reason and intuition behind defining the projective space in this way is that we can think of a projective space $\mathbb{P}^n$ as the regular vector space F^n adjoined with "points at infinity". Consider the example $\mathbb{P}^2_{\mathbb{R}}$: this contains points of the form $[x, y, z]$ for $x, y, z \in \mathbb{R}$, not all zero, where for any $\lambda \in \mathbb{R}^\times$, $[x, y, z] = [\lambda x, \lambda y, \lambda z]$. When $z \neq 0$, we may WLOG set $z = 1$ to get all points of the form $[x, y, 1]$ for $x, y \in \mathbb{R}$, and these naturally correspond to the points (x, y) in the real plane $\mathbb{R}^2$. But when $z = 0$, we get a family of points $[x, y, 0]$ whose third coordinate is zero, and these correspond exactly to the points at infinity along the direction of the vector (x, y) in $\mathbb{R}^2$.

Now a polynomial $f \in F[z_0, z_1, \ldots, z_n]$ on the vector space F^{n+1} in general does not define a function on $\mathbb{P}^n$, because there is no reason for f to be constant along a line, and in general it will not be. However, if f happens to be homogenous of degree $d \in \mathbb{N}$, then because

$$f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n),$$

it does make sense to talk about the zero locus of the polynomial f in $\mathbb{P}^n$.

Definition 2.2. A projective variety $X \subseteq \mathbb{P}^n$ is defined to be the zero locus of a family of homogenous polynomials $\{f_\alpha\}_{\alpha \in A}$ indexed by some set A.

We now discuss a concrete example in the case of $\mathbb{P}^2_{\mathbb{R}}$, i.e. $n = 2$ and $F = \mathbb{R}$. For any polynomial, $f \in F[x, y]$, we define the homogenization of the polynomial f by the homogenous polynomial $f_h \in F[x, y, z]$ that we get by simply mutliplying any terms of degree less than $\deg f$ by the required number of z's. Notice that on setting $z = 1$, we get back our original polynomial f, and all the solutions to $f_h|_{z=1} = 0$ are just the solutions to $f = 0$ in $\mathbb{R}^2$. However, when we set $z = 0$, we get the points at infinity along the curve $f = 0$ in $\mathbb{R}^2$.

Example 2.1. Let $f(x, y) = x^2 - y^2 - 1$. Then we get that $f_h(x, y, z) = x^2 - y^2 - z^2$. The solutions to $f_h(x, y, 1) = 0$ give us back the "real" points of the rectangular hyperbola $x^2 - y^2 = 1$. However, when we plug in $z = 0$, we get two additional points $[1, 1, 0]$ and $[1, -1, 0]$, which are exactly the two points at infinity along the hyperbola (the ones along the asymptotes $x - y = 0$ and $x + y = 0$ respectively).

For any $k \leq n$, we get the natural inclusion $F^k \hookrightarrow F^n$, and this induces the map $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{n-1}$. The image Σ of the inclusion is called a k-subspace or k-plane in F^n. If $k = 1$, we call Σ a line, and if $k = n - 1$, we call Σ a hyperplane. Observe that any such Σ can also be described as a zero locus of $n - k$ homogenous linear forms on $\mathbb{P}^{n-1}$, and is hence a subvariety of $\mathbb{P}^{n-1}$.

3 The Grassmannian

The set of k-planes in an n-dimensional vector space V can be seen, by the previous discussion, as a projective variety, and forms the central topic of the present discussion.

Definition 3.1. The Grassmannian $\mathrm{Gr}(k, V) := \{\Sigma \subseteq V : \dim \Sigma = k\}$ is the set of k-dimensional vector subspaces of V. Since the structure of $\mathrm{Gr}(k, V)$ depends only, upto isomorphism, on the dimension of V, we define $\mathrm{Gr}(k, n) := \{\Sigma \subseteq F^n : \dim \Sigma = k\}$.

Since k dimensional vector subspaces of F^n correspond exactly to $k-1$ dimensional subspaces of $\mathbb{P}^{n-1}$, the Grassmannian can be equivalently thought of as the set of $k-1$-planes in $n-1$-dimensional projective space; when we think of it in this way, the notation $\mathbb{G}(k-1, n-1)$ is customary.

To develop some intuition for $\mathrm{Gr}(k, n)$, we discuss some of its elementary properties.

3.1 Elementary Properties of the Grassmannian

1. $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k, n)$, with the canonical isomorphism being the complement map in F^n.
2. By definiton, for n-dimensional vector space V, $\mathbb{P}^{n-1} = \mathbb{P}V = \mathrm{Gr}(1, n)$.
3. The Grassmannian comes endowed with the action of $\mathrm{GL}_n(F)$ simply by the map $g\Sigma \mapsto \mathrm{im}(g|_\Sigma)$ for $g \in \mathrm{GL}_n(F)$ and $\Sigma \in \mathrm{Gr}(k, n)$.

This gives us a basic understanding of the structure of the Grassmannian. But how do we actually refer to individual elements of $\mathrm{Gr}(k, n)$?

4 The Plücker Embedding and Plücker Coordinates

Definition 4.1 (Plücker Embedding)**.** The Grassmannian $\mathrm{Gr}(k, V)$ comes equipped with an embedding $\Phi : \mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\bigwedge^k V)$ defined as follows: if $\Sigma \in \mathrm{Gr}(k, V)$ is spanned by the linearly independent $v_1, v_2, \ldots, v_k \in V$, then $\Phi(\Sigma) = [v_1 \wedge v_2 \wedge \cdots \wedge v_k]$.

Some things need to be unpacked before this can be seen that Φ actually defines a function. Firstly observe that this definition is independent of the choice of basis for Σ: if we had chosen a different basis, the wedge product $v_1 \wedge \cdots \wedge v_k$ would just get multiplied by the determinant of the change of basis matrix. Since this determinant is nonzero and elements of $\mathbb{P}(\bigwedge^k V)$ are only defined upto scalars, we do get a valid function. Now, to call it an "embedding", this function must be injective, and it is: given $\Phi(\Sigma)$, we can recover $\Sigma = \{v \in V \mid v \wedge \Phi(\Sigma) = 0\}$. Therefore, this is indeed a well-defined embedding.

Notice that this embedding is not in general surjective. For starters, the dimensions do not match up. We know that the "dimension" of $\mathbb{P}(\bigwedge^k V)$ for n-dimensional V is $\binom{n}{k} - 1$. (We use quotes because the projective space is not a vector space in the usual sense, but rather has $\binom{n}{k} - 1$ degrees of freedom.) The Grassmannian, on the other hand, as we shall show shortly, has "dimension" $k(n-k)$, which is, in general, much smaller. However, the image of this map $\Phi(\mathrm{Gr}(k, V))$ forms a subvariety of $\mathbb{P}(\bigwedge^k V)$ which can be described as the zero locus of certain homogenous relations in the coordinates of $\mathbb{P}(\bigwedge^k V)$ called the Plücker relations; these relations come from noticing that the only elements $[w] \in \mathbb{P}(\bigwedge^k V)$

that are elements of $\mathrm{Gr}(k,V)$ are the "totally decomposable" wedge products. A complete description of the Plücker coordinates is beyond the scope of this paper, and can be found in Manivel[2].

We now turn our attention to a more concrete way of talking about an element of the Grassmannian. For this, pick your favorite basis $E = \{e_1, e_2, \ldots, e_n\}$ for $V \cong F^n$, and write the vectors $v_1, v_2, \ldots, v_k$ that span $\Sigma \in \mathrm{Gr}(k,n)$ in terms of this basis. Then, to $\Sigma \in \mathrm{Gr}(k,n)$, we can associate a full rank $k \times n$ matrix, which we denote by M_Σ, whose rows form the coordinates of the vectors $v_1, \ldots, v_k$ with respect to the chosen basis E. Notice that this matrix representation is not unique, and is only defined modulo the equivalence relation of left multiplication by a matrix in $\mathrm{GL}_k(F)$. We now get a way to actually talk about elements of $\mathrm{Gr}(k,n)$ in terms of certain coordinates in $\mathbb{P}(\bigwedge^k V)$.

Definition 4.2 (Plücker Coordinates)**.** For a multi-index $I \in \binom{[n]}{k}$, define the I^{th} Plücker coordinate $P_I(\Sigma)$ as the maximal minor of M_Σ we get by choosing the k columns corresponding to the elements in I.

Example 4.1. Consider the $\Sigma \in \mathrm{Gr}(2,3)$ given by $M_\Sigma = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. In this case, the Plücker coordinates are indexed by $I \in \binom{[3]}{2} = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$. For convenience, to denote an element of $\binom{[3]}{2}$, we juxtapose the elements in it so that we can write $\binom{[3]}{2} = \{12, 13, 23\}$. The Plücker coordinates of Σ are then given by the determinants of the maximal minors of the form $P_{12}(\Sigma) = \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = 2$, $P_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$ and $P_{23} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$; and we then write $P(\Sigma) = [2,4,5]$. Notice again that these coordinates are only defined upto scalar multiples because left multiplication by an element of $\mathrm{GL}_2(\mathbb{R})$ will just scale all the Plücker coordinates by the same non-zero scalar which is its determinant.

Observe that the choice of basis $\{e_1, \ldots, e_n\}$ for V, gives a natural basis for $\bigwedge^k V$ by the elements $\bigwedge^k E := \{\bigwedge_{i \in I} e_i : I \in \binom{[n]}{k}\}$; then if we express the vectors $v_1, v_2, \ldots, v_k$ that span Σ in terms of the basis vectors $\{e_1, e_2, \ldots, e_n\}$, then the Plücker coordinates of Σ are simply the homogenous coordinates of $\Phi(\Sigma) = [v_1 \wedge \cdots \wedge v_k]$ in $\mathbb{P}(\bigwedge^k V)$ with respect to the basis $\bigwedge^k E$. In other words, the Plücker coordinates give us explicitly the Plücker embedding $\Phi : \mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \cong \mathbb{P}(\bigwedge^k V)$. This gives us a nice characterization of any arbitrary $k \times (n-k)$ matrix that tells us if it represents a Grassmannian.

Proposition 4.1. Let M be any $k \times n$ matrix. Then the following are equivalent:
1. M represents some $\Sigma \in \mathrm{Gr}(k,n)$.
2. The rows of M are linearly independent.
3. M has at least one nonzero maximal minor.

Proof. The double implication $1 \iff 2$ is clear. Now assume that the rows of M are not linearly independent; then rows of each of maximal submatrices of M cannot be linearly independent either, so that all the maximal minors are zero. Therefore $3 \implies 2$. From the above discussion, an element of the Grassmannian Σ gives an element $\Phi(\Sigma) = \mathbb{P}(\bigwedge^k V)$ having the Plücker coordinates of Σ, and the homogenous coordinates $[0, 0, \ldots, 0]$ do not represent a valid element of $\mathbb{P}(\bigwedge^k V)$. Therefore, $1 \implies 3$. This completes the proof. $\square$

Now that we know that the matrix representative of an element of the Grassmannian is not unique, is there some restriction we can impose on the matrix that will make this representative unique? As we shall soon see, the answer to this question leads us to some important insights.

The following section has been essentially taken from Billey[3].

Definition 4.3 (Row-Echelon Form)**.** Given a matrix M, it is said to be in row-echelon form if the following hold:

1. The rightmost entry of each row, called the "hook," is 1.
2. For each i, the hook of row $i+1$ is to the right of the hook of row i.
3. All entries above and below a hook are zero.

We know that we can always reduce a matrix to row-echelon form by elementary row operations, which correspond to left multiplication by invertible matrices–the algorithm basically involves taking the rightmost non-zero element of the lowest unchanged row, scaling it to one and making everything above and below it zero. For instance, the matrix $\begin{pmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{pmatrix}$, when written in row-echelon form is $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{pmatrix}$, because

$$\begin{pmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 1. *Every $\Sigma \in \mathrm{Gr}(k, n)$ can be expressed uniquely as a full rank $k \times n$ matrix M_Σ in row-echelon form.*

It is clear that such form is unique. Therefore, when we have a full rank $k \times n$ matrix, in its row-echelon form, we will be able to extract a $k \times k$ identity matrix from it; and WLOG, we may permute the basis elements to throw this $k \times k$ identity matrix to the extreme right; this leaves us essentially with a $k \times (n-k)$ matrix that is subject to no other conditions; that is we get $k(n-k)$ degrees of freedom to determine Σ. This means that the "dimension" of $\mathrm{Gr}(k, n)$ is $k(n-k)$ as promised.

For instance, from the following matrix representative of a $\Sigma \in \mathrm{Gr}(3, 5)$:

$$M_\Sigma = \begin{pmatrix} 9 & 1 & 0 & 0 & 0 \\ 8 & 0 & 7 & 1 & 0 \\ 6 & 0 & 2 & 0 & 1 \end{pmatrix},$$

picking out the columns 2, 4 and 5 containing the hooks gives us the 3×3 identity matrix. This motivates us to define a position map as follows:

Definition 4.4. Define the position map $\mathrm{pos} : \mathrm{Gr}(k,n) \to \binom{[n]}{k}$ simply by the positions of the hooks in the matrix representative when written in row-echelon form.

For instance, for the above example, we would get that $\mathrm{pos}(\Sigma) = \{2,4,5\}$. This then leads us to Schubert cells.

5 Schubert Cell Decomposition

This section deals with a decomposition of the Grassmannian into sets called Schubert cells named after German mathematician Hermann Schubert, who was one of the leading developers of the field of enumerative geometry.

Definition 5.1 (Schubert Cells)**.** Given a multi-index $I \in \binom{[n]}{k}$, define the Schubert cell $C_I = \{\Sigma \in \mathrm{Gr}(k,n) \mid \mathrm{pos}(\Sigma) = I\}$ as the elements of the Grassmannian that have hook positions I.

Example 5.1. In $\mathrm{Gr}(3,5)$, the Schubert cell $C_{\{2,4,5\}}$ consists of matrices whose hook positions are columns 2, 4 and 5. In other words,

$$C_{\{2,4,5\}} = \left\{ \begin{pmatrix} * & 1 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 \\ * & 0 & * & 0 & 1 \end{pmatrix} \right\},$$

where the $*$'s stand for undetermined entries. It is clear from this example that the dimension of $C_{\{2,4,5\}} = 5 = (2-1) + (4-2) + (5-3)$.

In general, we get that the dimension of C_I is given by $\sum_{i=1}^{k}(I_i - i) = \sum_{i \in I} i - \frac{k(k+1)}{2}$.

Since by Theorem 1 every element Σ of Grassmannian $\mathrm{Gr}(n,k)$ can be represented uniquely by a full rank $k \times n$ matrix in row-echelon form, and every such matrix has *some* position, we get the Schubert cell decomposition of the Grassmannian.

Theorem 2 (Schubert Cell Decomposition of the Grassmannian)**.**

$$\mathrm{Gr}(k,n) = \bigcup_{I \in \binom{[n]}{k}} C_I$$

We make some concluding remarks in this section without going too deep. For $I \in \binom{[n]}{k}$, the Schubert variety X_I is defined as the closure $\bar{C}_I$ of the Schubert cell C_I under the Zariski topology. It can be shown (cf. [2]) that the Grassmannian $\mathrm{Gr}(k,n) = X_{\{n-k+1,\ldots,n-1,n\}}$ is a Schubert variety. This gives another way to see that $\dim \mathrm{Gr}(k,n) = \sum_{i=1}^{k}(n-k+i-i) = k(n-k)$.

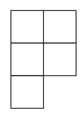
6 A Combinatorial Perspective

The Schubert cell decomposition of the Grassmannian has a very nice connection with the combinatorics of Young tableux: if you observe carefully, the stars in the above representation form a flipped Young tableu; that is no coincidence.

Theorem 3. *$\binom{[n]}{k}$ is in bijection with Young diagrams that fit in a $k \times (n-k)$ rectangle.*

Proof. It can be verified easily that the following map is a bijection: given a multi-index $I \in \binom{[n]}{k}$ containing elements $i_1 < i_2 < \cdots < i_k$, map I to the Young diagram of Y_I of shape given by $(i_k - k, i_{k-1} - (k-1), \ldots, i_1 - 1)$. $\square$

For example, the element $I = \{2 < 4 < 5\} \in \binom{[5]}{3}$ maps to the Young diagram of shape $(5-3, 4-2, 2-1) = (2, 2, 1)$, i.e. the Young diagram $Y_I \subseteq 3 \times 2$ that looks like:



This means that any element of the Schubert cell C_I is in bijection with some filling of the Young diagram Y_I, i.e. a Young tableu of shape Y_I. Further, this means that the Schubert cells can equivalently be indexed by Young diagrams $Y \subseteq k \times (n-k)$ that fit in a $k \times (n-k)$ rectangle. The dimension of such a Schubert cell is clearly $|Y|$. Then the Schubert cell decomposition has the form:

Theorem 4 (Schubert Cell Decomposition in terms of Young Diagrams)**.**

$$\mathrm{Gr}(k, n) = \bigcup_{Y \subseteq k \times (n-k)} C_Y$$

References

[1] Harris, Joe. *Algebraic Geometry: A First Course.* Springer New York : Springer, 1992.

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[3] Billey, Sara. *Tutorial on Schubert Varieties and Schubert Calculus.* ICERM Tutorials, Febrary 27, 013. `https://icerm.brown.edu/materials/Slides/sp-s13-off_weeks/Schubert_varieties_and_Schubert_calculus_]_Sara_Billey,_University_of_Washington.pdf`.