An Introduction to Modular Jacobians

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Abstract

We talk about Jacobians of compact Riemann surfaces, and then apply the theory to modular curves to deduce that the Hecke algebra over the integers is finitely generated, and to explain what the number field of a normalized eigenform is. We then briefly touch on how this relates to abelian varieties and modularity.

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1 Introduction to Jacobians

1.1 First Definitions

Let X be a compact Riemann surface. By the classification theorem for compact orientable 2-manifolds without boundary, X is a sphere with g handles for some integer $g \geq 0$, the genus of X. For $g \geq 1$, we describe the standard identified polygonal representation of X. Let Δ be a polygon of $4g$ sides, labeled in counterclockwise order as a_i, b_i, a_i', b_i' for $i = 1, \ldots, g$. Direct these a_i, b_i counterclockwise and a_i', b_i' clockwise. Then X is isomorphic to the quotient of Δ by identifying the sides a_i with a_i' and b_i with b_i' according to the directions specified. Since all the vertices of Δ are identified to a single point, the a_i and b_i become piecewise $\mathscr{C}^\infty$ cycles in X. Then it is a theorem in homology that:

$$H_1(X;\mathbf{Z}) = \bigoplus_{i=1}^{g} \mathbf{Z}[a_i] \oplus \bigoplus_{i=1}^{g} \mathbf{Z}[b_i] \cong \mathbf{Z}^{2g}.$$

(If $g = 0$, the direct sums above are understood to be empty.) For a proof, see any book on algebraic topology, for instance [1] Chapters 10 and 13 or [2] §2.2.

The following discussion is based on [3] Chapter VIII §1, 4, and [4] Chapters 14 and 15. If σ is a closed $\mathscr{C}^\infty$ 1-form on X and $D \subseteq X$ has a piecewise $\mathscr{C}^\infty$ boundary ∂D, then by Stokes' Theorem,

$$\int_{\partial D} \sigma = \iint_D \mathrm{d}\sigma = \iint_D 0 = 0,$$

so that the integral of a closed form around a boundary is zero. This means that the integral of a closed form along a cycle depends only on the homology class of the cycle. Now let $\Omega^1(X)$ denote the space of holomorphic differential 1-forms on X. It follows from the Cauchy-Riemann equations that every holomorphic 1-form $\omega \in \Omega^1(X)$ is closed. Since $H_1(X;\mathbf{Z})$ is generated by piecewise $\mathscr{C}^\infty$ cycles $\{[a_i],[b_i]\}$, we get a bilinear pairing:

$$\Omega^1(X) \times H_1(X;\mathbf{Z}) \to \mathbf{C}, \quad (\omega, [\gamma]) \mapsto \int_\gamma \omega.$$

This induces a map $\int_{-} : H_1(X;\mathbf{Z}) \to \Omega^1(X)^\vee$. Let

$$\Pi := \int_{-} H_1(X;\mathbf{Z}) \subseteq \Omega^1(X)^\vee$$

denote the image of this map; any element of Π is called a *period*.

Definition 1.1.1. Let X be a compact Riemann surface. The *Jacobian* of X is defined as the quotient group $\mathrm{Jac}(X) := \Omega^1(X)^\vee/\Pi$ of functionals on the space of holomorphic 1-forms on X modulo the periods.

In this paper, we build the theory of Jacobians of compact Riemann surfaces, and then specialize to the case of modular curves to deduce nontrivial facts about the algebra of Hecke and diamond operators by interpreting them geometrically.

Example 1.1.2. Let $X = \mathbf{P}^1_{\mathbf{C}}$ be the Riemann sphere, so that $g = 0$. We show that $\Omega^1(X)$ is trivial. Let $\omega \in \Omega^1(X)$, and write $\omega = f(z)\mathrm{d}z = -w^{-2} f(1/w)\mathrm{d}w$. Since every meromorphic function on X is rational (c.f. [3] Chapter II Theorem 2.1), f is a rational function. Since we want it to have no poles in $\mathbf{C}$, f must be polynomial, but then ω has a pole at ∞ of order at least two, unless $\omega \equiv 0$. This shows that $\Omega^1(X)$ is trivial, and therefore so is $\mathrm{Jac}(X)$.

Note that for any compact Riemann surface X, the Jacobian $\mathrm{Jac}(X)$ is an abelian group. It is trivial in the case $g = 0$, but as we shall see in the next section, for $g \geq 1$, $\mathrm{Jac}(X)$ is nontrivial and we have a much better description of $\mathrm{Jac}(X)$ as a g-dimensional complex torus.

1.2 Integration is Perfect

Assume now that $g \geq 1$. For any σ a closed $\mathscr{C}^\infty$ 1-form on X and $i = 1, \ldots, g$, let

$$A_i(\sigma) := \int_{[a_i]} \sigma \text{ and } B_i(\sigma) := \int_{[b_i]} \sigma.$$

For a fixed σ, these $2g$ numbers care called the *periods* of σ. Now, let $\pi : \Delta \to X$ denote the projection map; note that this is $1-1$ away from $\partial\Delta$. Then we can pull any σ on X back to $\pi^*\sigma$ on Δ. Since pullbacks commute with the exterior derivative, we have

$$\mathrm{d}\pi^*\sigma = \pi^*\mathrm{d}\sigma = \pi^*0 = 0,$$

so that $\pi^*\sigma$ is closed. Choose a base point $p_0 \in \Delta^\circ$ and for $p \in \Delta$ define:

$$f_\sigma(p) := \int_{p_0}^{p} \pi^*\sigma,$$

where the integral is taken over any path that lies completely in Δ. Since $\pi_1(\Delta) = 0$ and $\pi^*\sigma$ is closed, this map is well-defined and independent of choice of path. Note that by the Fundamental Theorem of Calculus, we have that $\mathrm{d}f_\sigma = \pi^*\sigma$. In particular, f_σ is holomorphic if σ is. Then we have the following fundamental result:

Lemma 1.2.1. Let σ, τ be closed $\mathscr{C}^\infty$ 1-forms on X. Then $\displaystyle\int_{\partial\Delta} f_\sigma \pi^*\tau = \sum_{i=1}^{g} A_i(\sigma)B_i(\tau) - A_i(\tau)B_i(\sigma)$.

Proof. For any $p \in a_i$ and $p' \in a_i'$ corresponding to the same point on X, let α_p denote a path in Δ from p to p'. Then α_p is homotopic to b_i, so that

$$f_\sigma(p) - f_\sigma(p') = \int_{p_0}^{p} \pi^*\sigma - \int_{p_0}^{p'} \pi^*\sigma = -\int_{\alpha_p} \pi^*\sigma = -\int_{b_i} \pi^*\sigma = -B_i(\sigma).$$

Similarly, for any $q \in b_i$ and $q' \in b_i'$, if β_q is a path from q to q', then β_q is homotopic to a_i'. But now, by the change of variables formula, we have that

$$\int_{a_i'} \pi^*\sigma = \int_{\pi(a_i')} \sigma = \int_{-\pi(a_i)} \sigma = -\int_{\pi(a_i)} \sigma = -\int_{a_i} \pi^*\sigma,$$

so that

$$f_\sigma(q) - f_\sigma(q') = -\int_{\beta_q} \pi^*\sigma = -\int_{a_i'} \pi^*\sigma = \int_{a_i} \pi^*\sigma = A_i(\sigma).$$

Since τ is a 1-form on X, the values of $\pi^*\tau$ along a_i and a_i' are equal, so that:

$$\begin{aligned}
\int_{\partial\Delta} f_\sigma \pi^*\tau &= \sum_{i=1}^{g} \left(\int_{a_i} - \int_{a_i'} + \int_{b_i} - \int_{b_i'} \right) f_\sigma \pi^*\tau \\
&= \sum_{i=1}^{g} \int_{p \in a_i} \left(f_\sigma(p) - f_\sigma(p') \right) \pi^*\tau + \int_{q \in b_i} \left(f_\sigma(q) - f_\sigma(q') \right) \pi^*\tau \\
&= \sum_{i=1}^{g} -B_i(\sigma)A_i(\tau) + A_i(\sigma)B_i(\tau).
\end{aligned}$$

■

Corollary 1.2.2. If $0 \neq \omega \in \Omega^1(X)$, then $\operatorname{Im} \sum_{i=1}^{g} A_i(\omega)\overline{B_i(\omega)} < 0$. In particular, if $\omega \in \Omega^1(X)$ such that $A_i(\omega) = 0$ for all $i = 1, \ldots, g$, then $\omega = 0$. The same conclusion holds if $B_i(\omega) = 0$ for all $i = 1, \ldots, g$.

Proof. Since ω is holomorphic, $\mathrm{d}\omega = 0$ so that $\mathrm{d}\overline{\omega} = 0$ too. By Stokes' Theorem,

$$\int_{\partial\Delta} f_\omega \pi^*\overline{\omega} = \iint_\Delta \mathrm{d}(f_\omega \pi^*\overline{\omega}) = \iint_\Delta \pi^*\omega \wedge \pi^*\overline{\omega} + f_\omega \wedge \pi^*\mathrm{d}\overline{\omega} = \iint_\Delta \pi^*(\omega \wedge \overline{\omega}) = \iint_X \omega \wedge \overline{\omega}.$$

Locally, write $\omega = f(z)\mathrm{d}z$, so that $\overline{\omega} = \overline{f(z)}\mathrm{d}\bar{z}$. Then $\omega \wedge \overline{\omega} = |f|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z} = -2i|f|^2 \mathrm{d}x \wedge \mathrm{d}y$, so that

$$\operatorname{Im} \int_{\partial\Delta} f_\omega \pi^*\overline{\omega} = \operatorname{Im} \iint_X \omega \wedge \overline{\omega} < 0.$$

Therefore, using the lemma and that $A_i(\overline{\omega}) = \overline{A_i(\omega)}$ and likewise for B_i,

$$\operatorname{Im} \sum_{i=1}^g A_i(\omega)\overline{B_i(\omega)} = \frac{1}{2} \operatorname{Im} \sum_{i=1}^g A_i(\omega)B_i(\overline{\omega}) - A_i(\overline{\omega})B_i(\omega) = \frac{1}{2} \operatorname{Im} \int_{\partial\Delta} f_\omega \pi^*\overline{\omega} < 0.$$

■

Using the Riemann-Roch Theorem, it can be shown that $\dim_{\mathbf{C}} \Omega^1(X) = g$ (see [3] Chapter VI §3). (Note that this also proves immediately that $\Omega^1(\mathbf{P}^1_{\mathbf{C}})$ and hence $\mathrm{Jac}(\mathbf{P}^1_{\mathbf{C}})$ is trivial.) Let $\omega_1, \ldots, \omega_g$ be any basis for $\Omega^1(X)$ over $\mathbf{C}$. Define the *period matrices*

$$A = [A_i(\omega_j)]_{g \times g} \text{ and } B := [B_i(\omega_j)]_{g \times g}.$$

Note that while the basis is not specified in the notation, these depend on the choice of basis of $\Omega^1(X)$. Then from the above, we have that:

Corollary 1.2.3. The period matrices are nonsingular. Further, $A^\top B = B^\top A$.

Proof. Suppose that $c = [c_1, c_2, \ldots, c_g]^\top$ is such that $Ac = 0$. Let $\omega = \sum_{j=1}^g c_j \omega_j$, then $A_i(\omega) = \sum_{j=1}^g c_j A_i \omega_j = 0$ for each $i = 1, \ldots, g$. By the above corollary, this means that $\omega = 0$, so that $c_j = 0$ for all $j = 1, \ldots, g$. The proof for B is the same. This means that the period matrices are nonsingular.

To prove the second part, apply the lemma to $\sigma = \omega_j$ and $\tau = \omega_k$ for $1 \le j,k \le g$. Since ω_j and ω_k both have type $(1,0)$, we have that $\omega_j \wedge \omega_k = 0$, and $\mathrm{d}\omega_k = 0$. Therefore,

$$\int_{\partial\Delta} f_{\omega_j} \pi^*\omega_k = \iint_\Delta \mathrm{d}\big(f_{\omega_j} \pi^*\omega_k\big) = \iint_\Delta \pi^*(\omega_j \wedge \omega_k) + f_{\omega_j} \wedge \pi^*\mathrm{d}\omega_k = 0.$$

By the lemma, this means that

$$[A^\top B]_{j,k} = \sum_{i=1}^g A_i(\omega_j)B_i(\omega_k) = \sum_{i=1}^g A_i(\omega_k)B_i(\omega_j) = [B^\top A]_{j,k}.$$

■

Since A is nonsingular, there is a basis $\{\omega_j\}$ such that the period matrix $A = \mathrm{id}_g$ (such a transformation can be achieved by a change of basis matrix with real coefficients). A basis of $\Omega^1(X)$ satisfying $A = \mathrm{id}_g$ is called *normalized* with respect to the generators $\{[a_i], [b_j]\}$ of $H_1(X;\mathbf{Z})$. In this case, the other period matrix B satisfies some symmetry conditions, as first shown by Riemann.

Lemma 1.2.4 (Riemann's Bilinear Relations)**.** If B is the b-period matrix with respect to a normalized basis of $\Omega^1(X)$ (i.e. such that $A = \mathrm{id}_g$), then B is symmetric and has positive definite imaginary part.

Proof. Let $\{\omega_1, \ldots, \omega_g\}$ be a normalized basis. Since $A = \mathrm{id}_g$, symmetry of B follows from Corollary 1.2.3. Let $c_1, \ldots, c_g \in \mathbf{R}$ be not all zero, and let $\omega = \sum_{j=1}^g c_j \omega_j$. By Corollary 1.2.2,

$$\operatorname{Im} \sum_{i=1}^g A_i(\omega)\overline{B_i(\omega)} < 0.$$

But $A_i(\omega) = c_i$, so that we have

$$\sum_{i=1}^{g} A_i(\omega)\overline{B_i(\omega)} = \sum_{i=1}^{g} c_i \sum_{j=1}^{g} \overline{c_j B_i(\omega_j)} = \sum_{i,j} c_i c_j \overline{B_i(\omega_j)},$$

because $c_j \in \mathbf{R}$. Therefore, by taking complex conjugates, we get that

$$c^\top \operatorname{Im}(B) c = \operatorname{Im}(c^\top B c) = \operatorname{Im} \sum_{i,j} c_i c_j B_i(\omega_j) > 0,$$

which means exactly that $\operatorname{Im} B$ is a positive definite real matrix. ■

Corollary 1.2.5. The $2g$ rows of the period matrices A and B are $\mathbf{R}$-linearly independent, and similarly for the $2g$ columns.

Proof. Assume WLOG that the basis is normalized. Suppose that there are row vectors $c, d \in \mathbf{R}^{1\times g}$ such that $\begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} \mathrm{id}_g \\ B \end{bmatrix} = 0$, i.e. $c + dB = 0$. Taking the imaginary part gives $d \operatorname{Im}(B) = 0$, so that by the above $d = 0$; and this implies $c = 0$. The proof for columns is analogous. ■

We are now ready to prove what we wanted:

Proposition 1.2.6. The bilinear pairing

$$\Omega^1(X) \times H_1(X;\mathbf{Z}) \to \mathbf{C}, \quad (\omega, [\gamma]) \mapsto \int_\gamma \omega$$

is a perfect pairing of $\mathbf{Z}$-modules.

Proof. We want to show that the pairing is nondegenerate. Suppose there is $\omega \in \Omega^1(X)$ is such that $\forall [\gamma] \in H_1(X;\mathbf{Z}) : \int_\gamma \omega = 0$. Then, in particular, for all $i = 1, \ldots, g : A_i(\omega) = 0$. By Corollary 1.2.2, this means that $\omega = 0$. Conversely, suppose $[\gamma] \in H_1(X;\mathbf{Z})$ is such that $\forall \omega \in \Omega^1(X) : \int_\gamma \omega = 0$. Write $[\gamma] = \sum_{i=1}^g c_i[a_i] + \sum_{i=1}^g d_i[b_i]$. Let c and d be the real $1 \times g$ row vectors with entries c_i and d_i respectively. By our assumption, we see that if A and B are period matrices, then $cA + dB = 0$. By Corollary 1.2.5, this means that $c = d = 0$, so that $[\gamma] = 0$. ■

This proposition tells us that $\Pi = \int_{-} H_1(X;\mathbf{Z}) \subseteq \Omega^1(X)^\vee$ is a full-rank lattice, i.e. $\Omega^1(X)^\vee \cong \Pi \otimes_{\mathbf{Z}} \mathbf{R}$, so that

$$\text{since } \Pi = \bigoplus_{i=1}^{g} \mathbf{Z} \int_{a_i} \oplus \bigoplus_{i=1}^{g} \mathbf{Z} \int_{b_i}, \text{ we have } \Omega^1(X)^\vee = \bigoplus_{i=1}^{g} \mathbf{R} \int_{a_i} \oplus \bigoplus_{i=1}^{g} \mathbf{R} \int_{b_i}.$$

More generally, we have the following definition:

Definition 1.2.7. Let $g \geq 1$. A *full-rank lattice* $\Pi \subseteq \mathbf{C}^g$ is a discrete additive subgroup of $\mathbf{C}^g$ satisfying $\Pi \otimes_{\mathbf{Z}} \mathbf{R} = \mathbf{C}^g$. The quotient $\mathbf{C}^g/\Pi$ of $\mathbf{C}^g$ by a full-rank lattice Π is called a *g-dimensional complex torus*.

The 0-dimensional complex torus is defined to be the trivial group. Observe that a g-dimensional complex torus is isomorphic to $(\mathcal{S}^1)^{2g}$, which is a compact real Lie group; i.e. a g-dimensional torus is a compact complex Lie group. For example, a 1-dimensional complex torus is simply an elliptic curve. In this terminology, we have shown:

Proposition 1.2.8. Let X be a compact Riemann surface of genus g. Then $\operatorname{Jac}(X) := \Omega^1(X)^\vee/\Pi$ is a g-dimensional complex torus. In particular, it is compact, and for $g \geq 1$, nontrivial.

We use the ideas of this section to compute the Jacobian of elliptic curves.

Proposition 1.2.9. Let $X = \mathbf{C}/\Lambda$ be a complex torus, i.e. an elliptic curve. Then $\mathrm{Jac}(X) \cong X$.

Proof. We know that for a complex torus, $g = 1$. Suppose that $\Lambda = \mathbf{Z}\langle \omega_1, \omega_2 \rangle$. Then by the discussion in the previous paragraph,

$$\Pi = \mathbf{Z}\int_0^{\omega_1} \oplus \quad \mathbf{Z}\int_0^{\omega_2}.$$

Since $\Omega^1(X) = \mathbf{C}\langle \mathrm{d}z \rangle$, the map $\Omega^1(X)^\vee \to \mathbf{C}$ given by $\lambda \mapsto \lambda(\mathrm{d}z)$ is an isomorphism. Under this isomorphism, the image of Π is generated by $\int_0^{\omega_1} \mathrm{d}z = \omega_1$ and $\int_0^{\omega_2} \mathrm{d}z = \omega_2$, i.e. the image of Π is simply Λ. Therefore, we see that

$$\mathrm{Jac}(X) := \Omega^1(X)^\vee/\Pi \cong \mathbf{C}/\Lambda = X.$$

■

The Jacobian and its quotients, called *abelian varieties*, are very important objects of study in the theory of modular forms. We will have more to say about them soon.

1.3 Abel-Jacobi Theorem

This section is based on [3] Chapter VII. Choose a base point $p_0 \in X$, and for each $p \in X$, choose a path γ_p in X from p_0 to p. Define the map $A : X \to \Omega^1(X)^\vee$ by

$$A(p)(\omega) := \int_{\gamma_p} \omega.$$

This map is not well-defined as it depends on the choice of path γ_p, but choosing a different path γ_p' would change the functional by integration about the cycle $\gamma_p - \gamma_p'$. Therefore, we get a well-defined map $A : X \to \mathrm{Jac}(X)$, which can then be extended by linearity to a group homomorphism $A : \mathrm{Div}(X) \to \mathrm{Jac}(X)$.

Definition 1.3.1. Fix a base point $p_0 \in X$. The map $A : \mathrm{Div}(X) \to \mathrm{Jac}(X)$ given by

$$\sum_p n_p(p) \mapsto \sum_p n_p \int_{p_0}^{p}$$

is a well-defined homomorphism, and is called the *Abel-Jacobi map* for X.

Recall that we have the short exact sequences:

$$0 \to \mathrm{Div}^0(X) \to \mathrm{Div}(X) \xrightarrow{\deg} \mathbf{Z} \to 0, \text{ and}$$

$$0 \to \mathbf{C}^* \to \mathbf{C}(X)^* \xrightarrow{\mathrm{div}} \mathrm{Div}^0(X) \to \mathrm{Pic}^0(X) \to 0.$$

Restricting A to $\mathrm{Div}^0(X)$ gives us the map $A_0 : \mathrm{Div}^0(X) \to \mathrm{Jac}(X)$. Why would we want to do that? Well, while the map A depends on the choice p_0 of basepoint, this restricted map A_0 does not!

Proposition 1.3.2. The restricted Abel-Jacobi map $A_0 : \mathrm{Div}^0(X) \to \mathrm{Jac}(X)$ is independent of the choice of base point p_0.

Proof. Suppose we choose a different base point p_0', and let γ be any path from p_0 to p_0' in X. Then the image of the Abel-Jacobi map changes by $\int_\gamma$, which is independent of the points $p \in X$. Therefore, we get that if $\sum_p n_p = 0$ then $A\left(\sum_p n_p(p)\right)$ changes by

$$\sum_p n_p \int_\gamma = \left(\sum_p n_p\right)\int_\gamma = 0.$$

■

We are now in a position to state the very important Abel-Jacobi Theorem.

Theorem 1.3.3 (Abel-Jacobi)**.** Let X be a compact Riemann surface, and let $D \in \mathrm{Div}^0(X)$. Then $D \in \mathrm{div}(\mathbf{C}(X)^*)$ iff $A_0(D) = 0 \in \mathrm{Jac}(X)$. Further, the map A_0 surjects onto $\mathrm{Jac}(X)$, so that it induces an isomorphism

$$\mathrm{Div}^0(X)/\mathrm{div}(\mathbf{C}(X)^*) = \mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X).$$

A partial proof can be found in [3] Chapter VII, and complete proofs can be found in [4] Chapter 15 and [5] Chapter III. We shall focus on some examples and applications of the theorem. In particular, in §2 of the paper, we will use many times the identification of $\mathrm{Pic}^0(X)$ with $\mathrm{Jac}(X)$.

Example 1.3.4. In the case $X = \mathbf{P}^1_{\mathbf{C}}$, we've seen that $\mathrm{Jac}(X) = 0$. The Abel-Jacobi Theorem tells us that $\mathrm{Pic}^0(X) = 0$, which is the familiar fact that every degree-0 divisor on the Riemann sphere is principal, i.e. if $D = \sum(z_i) - \sum(p_j) - n_\infty(\infty)$ has degree zero, then the rational function $\prod(z - z_i)/\prod(z - p_j)$ has divisor D.

Example 1.3.5. Suppose $X = \mathbf{C}/\Lambda$ is an elliptic curve, so that $g = 1$. Then by Proposition 1.2.9 and the Abel-Jacobi Theorem, we get the chain of isomorphisms:

$$\begin{array}{ccccc} \mathrm{Pic}^0(X) & \xrightarrow{\sim} & \mathrm{Jac}(X) & \xrightarrow{\qquad\sim\qquad} & X \\ \left[\sum_p n_p(p)\right] & \longmapsto & \sum\limits_p \int_{p_0}^{p} & \longmapsto & \sum\limits_p n_p \int_{p_0}^{p} \mathrm{d}z = \sum\limits_p n_p(p - p_0) = \sum\limits_p n_p p. \end{array}$$

In other words, it recovers the well-known condition that a divisor of degree zero $D = \sum_p n_p(p) \in \mathrm{Div}^0(X)$ on an elliptic curve X is the divisor of a function iff $\sum_p n_p p = 0 \in X$ in the group law.

1.4 Applications of the Abel-Jacobi Theorem

We use the Abel-Jacobi map to classify some Riemann surfaces of low genera. For that we use the following proposition:

Proposition 1.4.1. Let X be a compact Riemann surface of genus g. If the Abel-Jacobi map $A : X \to \mathrm{Jac}(X)$ is not injective, then $X \cong \mathbf{P}^1_{\mathbf{C}}$ and $g = 0$.

Proof. Suppose that $A(p) = A(q)$ for some $p \neq q$, i.e. $A((p) - (q)) = 0 \in \mathrm{Jac}(X)$. By composing with the isomorphism with $\mathrm{Pic}^0(X)$, $[(p) - (q)] = 0 \in \mathrm{Pic}^0(X)$. This means that there is a meromorphic $F \in \mathbf{C}(X)^*$ such that $(p) - (q) = \mathrm{div}(F)$, i.e. F is map with a simple zero at p, a simple pole at q and no other zeroes or poles. But then the associated holomorphic map of Riemann surfaces $F : X \to \mathbf{P}^1_{\mathbf{C}}$ has degree one, and is hence an isomorphism. ■

Corollary 1.4.2. If X is a compact Riemann surface of genus 0, then $X \cong \mathbf{P}^1_{\mathbf{C}}$ is the Riemann sphere.

Proof. If $g = 0$, then $\Omega^1(X)$ and hence $\mathrm{Jac}(X)$ is trivial, so that the map $A : X \to \mathrm{Jac}(X)$ cannot be injective. ■

Corollary 1.4.3. If X is a compact Riemann surface of genus $g \geq 1$, then the Abel-Jacobi map $A : X \to \mathrm{Jac}(X)$ is injective.

We now use the Abel-Jacobi Theorem to show that every compact Riemann surface X of genus 1 is an elliptic curve, i.e. of the form $\mathbf{C}/\Lambda$ for a full rank lattice $\Lambda \subseteq \mathbf{C}$.

Proposition 1.4.4. Suppose X is a compact Riemann surface of genus $g = 1$. Then X is isomorphic to a complex torus, i.e. there is a full rank lattice $\Lambda \subseteq \mathbf{C}$ such that $X \cong \mathbf{C}/\Lambda$. Further, given any point $p_0 \in X$, there is an isomorphism of X with the complex torus $\mathbf{C}/\Lambda$ such that $p_0 \mapsto 0$.

Proof. Since X has genus one, $\mathrm{Jac}(X) \cong \mathbf{C}/\Pi$ is a complex torus. More precisely, we have the isomorphism $\mathrm{Jac}(X) \xrightarrow{\sim} \mathbf{C}/\Pi$ given by $[\lambda] \mapsto \lambda(\omega)$ for some (any) $0 \neq \omega \in \Omega^1(X)$. Under this identification, the Abel-Jacobi map $A : X \to \mathrm{Jac}(X) \to \mathbf{C}/\Pi$ is given locally as $p \mapsto \int_{p_0}^{p} \omega$, and is therefore a holomorphic function of p. This means that the Abel-Jacobi map for a curve of genus 1 is an injective holomorphic map of compact Riemann surfaces, and is therefore an isomorphism. Further, the choice of p_0 is the only choice made in defining A. ∎

This theorem means that every genus-one curve is an abelian group, with group law induced by the Abel-Jacobi isomorphism. This can then be used to define a group law on a smooth projective plane cubic X: by Plücker's formula, such a cubic would have genus $(3-1)(3-2)/2 = 1$. By the above proposition, X is then isomorphic to a complex torus, and so it is an abelian group. Working out the details gives back the usual chordal definition of addition on an elliptic curve as a projective variety, and shows that it is a group without having to prove associativity using the Cayley-Bacharach theorem, for instance. For more in this direction, see [3] Chapter VII §5 or [6] Chapter III §3.

1.5 Maps of Jacobians, Divisors, and Picards

Suppose X and Y are compact Riemann surfaces, and $F : X \to Y$ is a nonconstant holomorphic map. Following [3] and [7], we define the forward and backward maps F_J, F^J of Jacobians, F_D, F^D of divisor groups, and F_P, F^P of Picard groups respectively in a manner compatible with restriction map and the Abel-Jacobi isomorphism.

To define the forward map, observe that F induces a pullback on holomorphic differentials. This gives a linear map $F^* : \Omega^1(Y) \to \Omega^1(X)$, which then gives rise to the dual map:

$$F_* := (F^*)^\vee : \Omega^1(X)^\vee \to \Omega^1(Y)^\vee.$$

For a path γ in X and $\omega \in \Omega^1(Y)$, the change of variables formula gives us that

$$\left(F_* \int_\gamma\right) \omega = \int_\gamma F^* \omega = \int_{F \circ \gamma} \omega.$$

In particular, if γ is a cycle, then so is $F \circ \gamma$, so that F_* takes homology to homology. This gives rise to a holomorphic homomorphism of Jacobians:

Definition 1.5.1. Suppose $F : X \to Y$. The forward map of Jacobians is given by

$$F_J : \mathrm{Jac}(X) \to \mathrm{Jac}(Y), \quad F_J[\lambda] = [F_* \lambda] = [\lambda \circ F^*].$$

This map is holomorphic since it descends from a linear map on the original complex vector spaces. By the Abel-Jacobi isomorphism $\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X)$, every element of $\mathrm{Jac}(X)$ can be written as $\left[\sum_p n_p \int_{p_0}^{p}\right]$; then the map F_J is given simply by

$$F_J \left[\sum_p n_p \int_{p_0}^{p}\right] = \left[\sum_p n_p \int_{F(p_0)}^{F(p)}\right].$$

We want to describe the corresponding forward map $F_P : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pic}^0(X) \arrow[r, "\sim"] \arrow[d, "F_P"] & \mathrm{Jac}(X) \arrow[d, "F_J"] \\ \mathrm{Pic}^0(Y) \arrow[r, "\sim"] & \mathrm{Jac}(Y). \end{tikzcd}$$

Therefore, we want F_P to take $\sum_p n_p(p)$ to $\sum_p n_p(F(p))$. For this, we define more generally:

Definition 1.5.2. Suppose $F : X \to Y$. The forward map of divisor groups is given by

$$F_D : \mathrm{Div}(X) \to \mathrm{Div}(Y), \quad F\left(\sum_p n_p(p)\right) = \sum_p n_p(F(p)).$$

This map clearly preserves Div^0, i.e. takes $\mathrm{Div}^0(X)$ to $\mathrm{Div}^0(Y)$. In fact, the commutativity of the above diagram implies that this map also takes $\mathrm{div}(\mathbf{C}(X)^*)$ to $\mathrm{div}(\mathbf{C}(Y))^*$.[1] Therefore, this map descends to a map of Picard groups.

Definition 1.5.3. Suppose $F : X \to Y$. The forward map of Picard groups is given by

$$F_P : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y), \quad F\left[\sum_p n_p(p)\right] = \left[\sum_p n_p(F(p))\right].$$

The reverse map of Jacobians is slighly trickier to define. For that, we first define the F-*trace* of a meromorphic 1-form ω on X. Away from branch points, F is a d-fold covering map (where $d := \deg F$), so that if $q \in Y$ is not a branch point, there is a chart domain U containing q such that $F^{-1}(U)$ is the disjoint union of d chart domains $V_1, \dots, V_d$ with $F|_{V_i}$ biholomorphic for each i. Let $F_i^{-1} : U \to V_i$ be the inverse of $F|_{V_i}$; then we define the F-trace of ω on U to be

$$\mathrm{Tr}_F(\omega) := \sum_{i=1}^{d} (F_i^{-1})^*(\omega|_{V_i}).$$

We quote some properties of this map:

Proposition 1.5.4. This definition of trace extends nicely to the branch points of Y, so that we get a well-defined $\mathbf{C}$-linear map of meromorphic differential 1-forms:

$$\mathrm{Tr}_F : \mathcal{M}^1(X) \to \mathcal{M}^1(Y).$$

Further, this map takes holomorphic forms to holomorphic forms, so that we get a linear map:

$$\mathrm{Tr}_F : \Omega^1(X) \to \Omega^1(Y).$$

For a proof, see [3] Chapter VII §3. As before, we dualize to get the map

$$\mathrm{Tr}_F^\vee : \Omega^1(Y)^\vee \to \Omega^1(X)^\vee.$$

We now want to show that this map takes homology to homology. Away from the branch locus, F is a d-fold covering map, so that given a path γ in Y, at all but finitely many paints of Y we may lift γ to exactly d preimages $\gamma_1, \dots, \gamma_d$. These paths come together at the ramification points, but in any case we may take the closure of these and obtain d lifts, which we again call γ_i.

[1] A more explicit way to see this, explained in [7], is to define a $\mathbf{C}$-linear map $\mathrm{N}_F : \mathbf{C}(X) \to \mathbf{C}(Y)$ called the *norm* of F, defined by $(\mathrm{N}_F\, g)(q) = \sum_{p \in F^{-1}(q)} g(p)^{e_p}$. It is not hard to see that with this definition, for any $g \in \mathbf{C}(X)^*$, we have that $F_D(\mathrm{div}(g)) = \mathrm{div}(\mathrm{N}_F\, g)$.

Definition 1.5.5. The pullback of a path γ in Y along F is the chain $F^*\gamma := \sum_{i=1}^d \gamma_i$.

With this definition, we have the following lemma.

Lemma 1.5.6. Let $F : X \to Y$. If γ is a path in Y and $\omega \in \Omega^1(X)$, then

$$\left(\mathrm{Tr}_F^\vee \int_\gamma\right)\omega = \int_\gamma \mathrm{Tr}_F(\omega) = \int_{F^*\gamma}\omega.$$

Proof. The integrals can be perturbed to avoid to the ramification and branch loci respectively, which by Cauchy's theorem doesn't affect the value of the integral of a holomorphic form. Effectively, therefore, the integrals don't "see" the ramification and brach loci respectively, and therefore we may assume that γ is a path not through any branch point. In this case, the left side is the sum of the integrals of ω along the lifts, and the right side is the integral of the sum of the appropriate ω's. By the change of variables formula, these are the same. ■

In particular, suppose that γ is a cycle in Y; and let $q \in \gamma$ be a point that is not a branch point. Then the map taking the initial point of each of the lifts of γ to its final point is a permutation of the d element set $F^{-1}(q)$, and so the lifts concatenate to loops in X corresponding to the permutation's cycles, i.e., $F^*\gamma$ is a cycle in X. This means exactly as required that $\mathrm{Tr}_F^\vee$ takes homology to homology, and we get a holomorphic homomorphism of Jacobians:

Definition 1.5.7. Suppose $F : X \to Y$. The backward map of Jacobians is given by

$$F^J : \mathrm{Jac}(Y) \to \mathrm{Jac}(X), \quad F^J[\lambda] = [\mathrm{Tr}_F^\vee \lambda] = [\lambda \circ \mathrm{Tr}_F].$$

Writing this explicity as before, the map F^J is given simply by

$$F^J\left[\sum_q n_q \int_{q_0}^q\right] = \left[\sum_q n_q \sum_{p\in F^{-1}(q)} e_p \int_{p_0}^p\right],$$

where $e_p = \mathrm{mult}_p(F)$ is the ramification index of p over q, i.e. the number of preimages of q that coincide at the single point p. As before, we want to describe the corresponding backward map F^P : $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Pic}^0(Y) \arrow[r, "\sim"] \arrow[d, "F^P"] & \mathrm{Jac}(Y) \arrow[d, "F^J"] \\
\mathrm{Pic}^0(X) \arrow[r, "\sim"] & \mathrm{Jac}(X).
\end{tikzcd}$$

Therefore, we want F^P to take $\sum_q n_q(q)$ to $\sum_q n_q \sum_{p\in F^{-1}(q)} e_p(p)$. For this, we define more generally:

Definition 1.5.8. Suppose $F : X \to Y$. The backward map of divisor groups is given by

$$F^D : \mathrm{Div}(Y) \to \mathrm{Div}(X), \quad F\left(\sum_q n_q(q)\right) = \sum_q n_q \sum_{p\in F^{-1}(q)} e_p(p).$$

Because for any $q \in Y$, $\sum_{p\in F^{-1}(q)} e_p = \deg(F)$ is constant, this map preserves Div^0 too. In fact, as before, the commutativity of the diagram implies that this map also takes $\mathrm{div}(\mathbf{C}(Y)^*)$ to $\mathrm{div}(\mathbf{C}(X)^*)$.[2]

[2] A more explicit way to see this is to observe that we have the pullback $F^* : \mathbf{C}(Y) \to \mathbf{C}(X)$, which satisfies for any $h \in \mathbf{C}(Y)$ and $p \in X$ that $\mathrm{ord}_p(F^*h) = e_p \,\mathrm{ord}_{F(p)}(h)$. Since F is nonconstant, it is surjective, and hence we see that

$$\mathrm{div}(F^*h) = \sum_p e_p \,\mathrm{ord}_{F(p)}(h)(p) = \sum_q \mathrm{ord}_q(h) \sum_{p\in F^{-1}(q)} e_p(p).$$

This means exactly that $F^D(\mathrm{div}(h)) = \mathrm{div}(F^*h)$.

Therefore, this map descends to a map of Picard groups.

Definition 1.5.9. Suppose $F : X \to Y$. The backward map of Picard groups is given by

$$F^P : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X), \quad F\left[\sum_q n_q(q)\right] = \left[\sum_q n_q \sum_{p\in F^{-1}(q)} e_p(p)\right].$$

We have therefore succesfully defined maps F_J, F_D, F_P and F^J, F^D, F^P corresponding to a map $F : X \to Y$ in a compatible manner.

2 Modular Jacobians

Following [7] Chapters 5 and 6, we now apply the developed theory to the special case when X is a modular curve. For that we begin with some general theory of modular forms and Hecke operators.

2.1 Modular Forms and Double Coset Operators

Definition 2.1.1. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup and $k \in \mathbf{Z}$. A holomorphic function $f : \mathfrak{h} \to \mathbf{C}$ is called a *modular form of weight k with respect to* Γ if

(a) f is weight-k invariant under the action of Γ, and
(b) $f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbf{Z})$.

The space of modular forms of weight k with respect to Γ is denoted by $\mathcal{M}_k(\Gamma)$. If in addition

(c) $a_0 = 0$ in the Fourier expansion of $f[\alpha]_k$ for all $\alpha \in \mathrm{SL}_2(\mathbf{Z})$,

then f is called a *cusp form*, and the space of cusp forms is denoted by $\mathcal{S}_k(\Gamma)$.

The conditions (2) and (3) need to be checked only for the (finitely many) coset representatives $\{\alpha_j\}$ in the decomposition $\mathrm{SL}_2(\mathbf{Z}) = \coprod_j \Gamma\alpha_j$. We need a small lemma:

Lemma 2.1.2. For any $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ and $f \in \mathcal{M}_k(\Gamma)$, $f[\gamma]_k$ has a Fourier expansion at ∞. If f is a cusp form, then the Fourier expansion of $f[\gamma]_k$ has constant term zero too.

Proof. We first claim that every $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ can be written as $\gamma = \theta\gamma'$, where $\theta \in \mathrm{SL}_2(\mathbf{Z})$ and $\gamma' = r\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $r \in \mathbf{Q}^+$ and $a,b,d \in \mathbf{Z}$ coprime and $a,d > 0$. To show this, the case $c = 0$ is clear, and if $c \neq 0$, then write $a/c = a'/c'$ for $a',c' \in \mathbf{Z}$ with $(a',c') = 1$; then the matrix $\theta \in \mathrm{SL}_2(\mathbf{Z})$ with bottom row $\begin{bmatrix} -c' & a' \end{bmatrix}$ works. By condition (2), we can assume therefore that $\gamma = \gamma'$ is of this form, but then if $f(\tau) = \sum_{n\geq 0} a_n q_h^n$, then

$$(f[\gamma]_k)(\tau) = r^{k-2}a^{k-1}d^{-1}f\left(\frac{a\tau+b}{d}\right) = r^{k-2}a^{k-1}d^{-1}\sum_{n\geq 0} a_n\zeta_{dh}^{bn}q_{dh}^{an},$$

where $\zeta_r = \exp(2\pi i/r)$ and $q_r = \exp(2\pi i\tau/r)$. The lemma follows. ∎

We use this to develop some more elaborate operators. Let $\Gamma_1, \Gamma_2 \leq \mathrm{SL}_2(\mathbf{Z})$ be congruence subgroups, and $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$. Then Γ_1 acts by left multiplcation on the double coset $\Gamma_1\alpha\Gamma_2 \subseteq \mathrm{GL}_2^+(\mathbf{Q})$, so that we may write $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2 = \coprod_j \Gamma_1\beta_j$ for some orbit representatives $\{\beta_j\}$. We first show that finitely many β_j suffice.

Lemma 2.1.3. If $\Gamma \leq \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup and $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, then so is $\alpha^{-1}\Gamma\alpha \cap \mathrm{SL}_2(\mathbf{Z})$.

Proof. Pick $\widetilde{N} \in \mathbf{Z}$ large enough so that simultaneously $\Gamma(\widetilde{N}) \subseteq \Gamma$, $\widetilde{N}\alpha \in \mathrm{Mat}_2(\mathbf{Z})$, and $\widetilde{N}\alpha^{-1} \in \mathrm{Mat}_2(\mathbf{Z})$, and let $N = \widetilde{N}^3$. Then

$$\alpha\Gamma(N)\alpha^{-1} \subseteq \alpha(\mathrm{id}_2 + \widetilde{N}^3\,\mathrm{Mat}_2(\mathbf{Z}))\alpha^{-1} = \mathrm{id}_2 + \widetilde{N}\cdot\widetilde{N}\alpha\cdot\mathrm{Mat}_2(\mathbf{Z})\cdot\widetilde{N}\alpha^{-1} \subseteq \mathrm{id}_2 + \widetilde{N}\,\mathrm{Mat}_2(\mathbf{Z}),$$

along with $\alpha\Gamma(N)\alpha^{-1} \leq \mathrm{SL}_2(\mathbf{Q})$, means $\alpha\Gamma(N)\alpha^{-1} \subseteq \Gamma(\widetilde{N}) \subseteq \Gamma$, so $\Gamma(N) \subseteq \alpha^{-1}\Gamma\alpha$. ∎

Lemma 2.1.4. Let $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 \subseteq \Gamma_2$. Then left multiplication by α induces map $\Gamma_2 \to \Gamma_1\alpha\Gamma_2$, which descends to an bijection:

$$\Gamma_3\backslash\Gamma_2 \xrightarrow{\sim} \Gamma_1\backslash\Gamma_1\alpha\Gamma_2.$$

In other words, $\{\gamma_{2,j}\}$ is a set of coset representatives for $\Gamma_3\backslash\Gamma_2$ iff $\{\beta_j\} = \{\alpha\gamma_{2,j}\}$ is a set of orbit representatives for $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. In particular, $\Gamma_3\backslash\Gamma_2$ is finite, and so is $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.

Proof. The map $\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ taking $\gamma_2 \mapsto \Gamma_1\alpha\gamma_2$ is surjective, and takes γ_2, γ_2' to the same orbit iff $\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2'$ i.e. $\gamma_2'\gamma_2^{-1} \in \Gamma_3$. The final statement follows since by Lemma 2.1.3, Γ_3 is a congruence subgroup, so that $[\Gamma_2 : \Gamma_3] \leq [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_3] < \infty$. ∎

This means that the following definition makes sense.

Definition 2.1.5. For congruence subgroups $\Gamma_1, \Gamma_2 \leq \mathrm{SL}_2(\mathbf{Z})$, $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, and $k \in \mathbf{Z}$ we define the *weight-k $\Gamma_1\alpha\Gamma_2$ operator* taking functions $f \in \mathcal{M}_k(\Gamma_1)$ to

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k,$$

where $\{\beta_j\}$ is a complete set of orbit representatives, i.e. s.t. $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2 = \coprod_j \Gamma_1\beta_j$.

Proposition 2.1.6. The double coset operator $[\Gamma_1\alpha\Gamma_2]_k$ is well-defined (irrespective of the choice of orbit representatives β_j), and takes $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$. Further, it takes cusp forms to cusp forms, i.e. $\mathcal{S}_k(\Gamma_1) \to \mathcal{S}_k(\Gamma_2)$.

Proof. First, $\Gamma_1\beta_j = \Gamma_1\beta_j' \Leftrightarrow \beta_j'\beta_j^{-1} \in \Gamma_1$. Since $f \in \mathcal{M}_k(\Gamma_1)$, we have$f[\beta_j'\beta_j^{-1}] = f$, so that $f[\beta_j'] = f[\beta_j]$, i.e. this operator is independent of the choice of orbit representatives. Next we show that $f[\Gamma_1\alpha\Gamma_2]$ is weight-k Γ_2-invariant. Any $\gamma_2 \in \Gamma_2$ induces by right multiplication a bijection $\gamma_2 : \Gamma_1\backslash\Gamma_1\alpha\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, so that if $\{\beta_j\}$ is a set of coset representatives, then so is $\{\beta_j\gamma_2\}$. Therefore,

$$f([\Gamma_1\alpha\Gamma_2]_k)[\gamma_2]_k = \sum_j f[\beta_j\gamma_2]_k = f[\Gamma_1\alpha\Gamma_2]_k.$$

Finally, we show that $f([\Gamma_1\alpha\Gamma_2])$ is holomorphic at cusps. For arbitrary $\delta \in \mathrm{SL}_2(\mathbf{Z})$, $f([\Gamma_1\alpha\Gamma_2]_k)[\delta]_k$ is a sum of a functions $f[\beta_j\delta]_k$, each of which is holomorphic at infinity by Lemma 2.1.2, and so it satisifies (2). This also shows that it preserves cusp forms. ∎

The following are some special cases of the double coset operator:

(a) If $\Gamma_1 \supseteq \Gamma_2$ and $\alpha = \mathrm{id}_2$, then $[\Gamma_1\alpha\Gamma_2]_k$ is the inclusion $\mathcal{M}_k(\Gamma_1) \hookrightarrow \mathcal{M}_k(\Gamma_2)$.
(b) If $\alpha^{-1}\Gamma_1\alpha = \Gamma_2$, then $f[\Gamma_1\alpha\Gamma_2]_k = f[\alpha]_k$, and the map $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$ is an isomorphism.
(c) If $\Gamma_1 \subseteq \Gamma_2$ and $\alpha = \mathrm{id}_2$, and $\{\gamma_{2,j}\}$ is a set of coset representatives of $\Gamma_1\backslash\Gamma_2$, then $f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\gamma_{2,j}]_k$ is the projection of $\mathcal{M}_k(\Gamma_1)$ onto the subspace $\mathcal{M}_k(\Gamma_2)$ by symmetrizing over the quotient. This map is clearly a surjection.

In general, any double coset operator is a composition of these three special cases. If $\Gamma_3' = \alpha\Gamma_3\alpha^{-1} = \Gamma_1 \cap \alpha\Gamma_2\alpha^{-1} \subseteq \Gamma_1$, then $\Gamma_1 \supseteq \Gamma_3'$, $\alpha^{-1}\Gamma_3'\alpha = \Gamma_3$ and $\Gamma_3 \subseteq \Gamma_2$, and by Lemma 2.1.4 the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_k(\Gamma_1) \arrow[r, hook, "{[\Gamma_1\,\mathrm{id}_2\,\Gamma_3']_k}"] \arrow[rrr, bend left, "{[\Gamma_1\alpha\Gamma_2]_k}"] & \mathcal{M}_k(\Gamma_3') \arrow[r, "{[\Gamma_3'\alpha\Gamma_3]_k}", "\sim"'] & \mathcal{M}_k(\Gamma_3) \arrow[r, two heads, "{[\Gamma_3\,\mathrm{id}_2\,\Gamma_2]_k}"] & \mathcal{M}_k(\Gamma_2) \\
f \arrow[r, mapsto] & f \arrow[r, mapsto] & f[\alpha]_k \arrow[r, mapsto] & \sum_j f[\alpha\gamma_{2,j}]_k = \sum_j f[\beta_j]_k.
\end{tikzcd}$$

In particular, if restrict this diagram to cusp forms, we obtain another commutative diagram, which we record here for reference.

$$\begin{tikzcd}[column sep=large]
\mathcal{S}_k(\Gamma_1) \arrow[r, hook, "{[\Gamma_1 \operatorname{id}_2 \Gamma_3']_k}"] \arrow[rrr, bend left, "{[\Gamma_1 \alpha \Gamma_2]_k}"] & \mathcal{S}_k(\Gamma_3') \arrow[r, "{[\Gamma_3' \alpha \Gamma_3]_k}", "\sim"'] & \mathcal{S}_k(\Gamma_3) \arrow[r, two heads, "{[\Gamma_3 \operatorname{id}_2 \Gamma_2]_k}"] & \mathcal{S}_k(\Gamma_2) \\
f \arrow[r, mapsto] & f \arrow[r, mapsto] & f[\alpha]_k \arrow[r, mapsto] & \sum_j f[\alpha\gamma_{2,j}]_k = \sum_j f[\beta_j]_k.
\end{tikzcd}$$

2.2 Hecke and Diamonds

Let's now define two important operators, the diamond and Hecke operators, as double coset operators. For this section, we follow [7],[8], and [9].

Recall that we have the surjection $\Gamma_0(N) \twoheadrightarrow (\mathbf{Z}/N\mathbf{Z})^*$ given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d$ with kernel $\Gamma_1(N)$, showing that $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^*$. Given any $\alpha \in \Gamma_0(N)$, consider the operator $[\Gamma_1(N)\alpha\Gamma_1(N)] : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$. Since $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$, this operator is case (2) from above i.e. simply $f[\alpha]_k$. This means that $\Gamma_0(N)$ acts on $\mathcal{M}_k(\Gamma_1(N))$; but the subgroup $\Gamma_1(N)$ acts trivially, so this is actually an action of the quotient $(\mathbf{Z}/N\mathbf{Z})^*$. This allows us to make the following definition.

Definition 2.2.1. For $\delta \in (\mathbf{Z}/N\mathbf{Z})^*$, define the *diamond operator* $\langle \delta \rangle$ by

$$\langle \delta \rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N)), \quad \langle \delta \rangle f = f[\alpha]_k \text{ for } \alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N), d \equiv \delta\,(N).$$

More generally, for $n \in \mathbf{N}$, define $\langle n \rangle$ by $\langle \bar{n} \rangle$ if $(n,N) = 1$ and 0 otherwise.

Definition 2.2.2. For $n \in \mathbf{N}$, define the *Hecke operator* $T_n : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ by

$$T_n := \sum_{\substack{ad=n \\ a|d \\ a,d>0}} \langle a \rangle \left[\Gamma_1(N) \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \Gamma_1(N) \right]_k.$$

For example, in the simple case $n = p$ a prime, this reduces to

$$T_p = \Gamma_1(N) \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \Gamma_1(N).$$

These operators satisfy many marvellous properties, some of which are summarized below.

Proposition 2.2.3. The Hecke and diamond operators T_n and $\langle n \rangle$ for any $n \geq 1$ preserve cusp forms, i.e. if $T = T_n$ or $T = \langle n \rangle$, then

$$T|_{\mathcal{S}_k(\Gamma_1)} : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N)).$$

Proof. All of these operators are (sums of) double coset operators, which by Proposition 2.1.6 preserve cusp forms. ∎

Proposition 2.2.4. The Hecke operators satisfy the two relations:

(a) $T_{nm} = T_n T_m$ if $(n,m) = 1$, and
(b) $T_{p^k} = T_p T_{p^{k-1}} - p^{k-1} \langle p \rangle T_{p^{k-2}}$ for $k \geq 2$.

Proposition 2.2.5. The Hecke and diamond operators commute, i.e. for every $m, n \geq 1$, we have that:

(a) $T_m T_n = T_n T_m$,

(b) $\langle n\rangle\langle m\rangle = \langle m\rangle\langle n\rangle$, and
(c) $T_n\langle m\rangle = \langle m\rangle T_n$.

This last proposition is best understood in terms of the Hecke algebra.

Definition 2.2.6. For $R = \mathbf{Z}$ or $\mathbf{C}$, define the *Hecke algebra over* R to be the algebra of endomorphisms of $\mathcal{M}_k(\Gamma_1(N))$ generated over R by the Hecke operators, i.e.

$$\mathbf{T}_R := R[\{T_n, \langle n\rangle : n \in \mathbf{N}] \subseteq \operatorname{End} \mathcal{M}_k(\Gamma_1(N)).$$

Each level N has its own Hecke algebra, but the N is usually omitted from the notation "since it is usually written somewhere nearby" ([7], p. 238). Then the above proposition can be restated as:

Proposition 2.2.7. The Hecke algebra $\mathbf{T}_R$ is a commutative unitary alegbra over R.

For proofs of propositions 10 and 11, see [9] Chapter 3.

Finally, the Hecke and diamond operators (at least for $(n,N) = 1$) are self-adjoint with respect to the Petersson inner product, so that they are diagonalizable. In particular, by commutativity, they are simultaneously diagonalizable, so that the following definition makes sense.

Definition 2.2.8. A $0 \neq f \in \mathcal{M}_k(\Gamma_1(N))$ that is a simultaneous eigenvector for all T_n and $\langle n\rangle$ for $n \geq 1$ is called an *eigenform*. An eigenform $f(\tau) = \sum_{n\geq 0} a_n q^n$ is said to be *normalized* if $a_1 = 1$.

Then we have the following amazing results:

Proposition 2.2.9. Suppose $k \geq 1$ and $f \in \mathcal{S}_k(\Gamma_1(N))$ is a normalized eigenform, i.e. its q-expansion is of the form

$$f(\tau) = q + \sum_{n\geq 2} a_n q^n.$$

Then for $n \geq 1, T_n f = a_n f$, i.e. the Fourier coefficients of f are its T_n eigenvalues.

For a proof, see [7] Theorem 5.8.2 or [10] Chapter III, Proposition 40. All of this is cute, but do such fanciful eigenforms even exist? They sure do!

Proposition 2.2.10. There is a subspace of $\mathcal{S}_k(\Gamma_1(N))$ which has a basis of normalized eigenforms.

This subspace of $\mathcal{S}_k(\Gamma_1(N))$ is the space $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ of new forms, the orthogonal complement of the space of old forms with respect to the Petersson inner product. For an explanation of what that is and the proof of this proposition, see [7] §5.8. We make the following definition:

Definition 2.2.11. A normalized eigenform $f \in \mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ is called a *newform*.

We are able to show one direction of the above proposition.

Proposition 2.2.12. The set of newforms in $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ is linearly independent.

Proof. Suppose there is a minimal nontrivial relation $\sum_{i=1}^m c_i f_i = 0$ with $c_i \in \mathbf{C}$ all nonzero; then $m \geq 2$. By proposition 2.2.9, for each $n \geq 1$, $T_n(f_i) = a_n(f_i)$. For any $n \geq 1$, applying $T_n - a_n(f_1)$ to the relation gives $\sum_{i=2}^m c_i(a_n(f_i) - a_n(f_1))f_i = 0$. By the minimality assumption, this must be trivial, so that $a_n(f_1) = a_n(f_i)$ for all i and n. But this means $f_i = f_1$ for all i, a contradiction since $m \geq 2$. ∎

We will make use of this definition and proposition later. An aim of the rest of the paper is to prove the following unexpected result:

Proposition 2.2.13. For $k = 2$ and any $N \geq 1$, the $\mathbf{Z}$-algebra $\mathbf{T}_{\mathbf{Z}} \subseteq \operatorname{End} \mathcal{S}_2(\Gamma_1(N))$, is finitely generated.

We will show this when we are able to appropriately interpret the action of double coset operators on the, wait for it, *Jacobians* of modular curves!

2.3 Action on Jacobians

In this section, we specialize to the case $k=2$, interpret the double coset maps geometrically, and see how they act on Jacobians. Recall that we have $\Gamma_1, \Gamma_2 \leq \mathrm{SL}_2(\mathbf{Z})$ congruence subgroups, $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 \subseteq \Gamma_2$ and $\Gamma_3' = \alpha\Gamma_3\alpha^{-1} = \Gamma_1 \cap \alpha\Gamma_2\alpha^{-1} \subseteq \Gamma_1$. Further, $\{\gamma_{2,j}\}$ denotes a set of coset representatives for $\Gamma_3\backslash\Gamma_2$, and $\{\beta_j\} = \{\alpha\gamma_{2,j}\}$ denotes a set of orbit representatives for $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. Finally, we have the maps

$$\Gamma_2 \hookleftarrow \Gamma_3 \xrightarrow{\sim} \Gamma_3' \hookrightarrow \Gamma_1,$$

where the middle map is the isomorphism $\gamma \mapsto \alpha\gamma\alpha^{-1}$. Letting $X_j := X(\Gamma_j) = \Gamma\backslash\mathfrak{h}^*$, we get the corresponding maps of modular curves

$$X_2 \xleftarrow{\pi_2} X_3 \xrightarrow{\alpha} X_3' \xrightarrow{\pi_1} X_1,$$

where the middle map is the curve isomorphism $\Gamma_3\tau \mapsto \Gamma_3'\alpha(\tau)$. Pulling an element $\Gamma_2\tau \in X_2$ across these to X_1 gives a map:

$$\Gamma_2\tau \xmapsto{\pi_2^{-1}} \{\Gamma_2\gamma_{2,j}(\tau)\} \xrightarrow{\alpha} \{\Gamma_3'\beta_j(\tau)\} \xrightarrow{\pi_1} \{\Gamma_1\beta_j(\tau)\},$$

which induces the map[3] $[\Gamma_1\alpha\Gamma_2]_2^\vee : \mathrm{Div}(X_2) \to \mathrm{Div}(X_1)$ that linearly extends the map given by $\Gamma_2\tau \mapsto \sum_j \Gamma_1\beta_j(\tau)$. In the language of §1.5, we now recognize this map to be

$$[\Gamma_1\alpha\Gamma_2]_2^\vee = \pi_{1,D} \circ \alpha_D \circ \pi_2^D : \mathrm{Div}(X_2) \to \mathrm{Div}(X_1),$$

which consequently descends to the map of Picard groups

$$[\Gamma_1\alpha\Gamma_2]_2^\vee = \pi_{1,P} \circ \alpha_P \circ \pi_2^P : \mathrm{Pic}^0(X_2) \to \mathrm{Pic}^0(X_1), \quad \left[\sum_\tau n_\tau \Gamma_2\tau\right] \mapsto \left[\sum_\tau n_\tau \sum_j \Gamma_1\beta_j(\tau)\right].$$

By the Abel-Jacobi isomorphism and the compatibility of the forward and backward maps, we therefore get the map

$$[\Gamma_1\alpha\Gamma_2]_2^\vee = \pi_{1,J} \circ \alpha_J \circ \pi_2^J : \mathrm{Jac}(X_2) \to \mathrm{Jac}(X_1), \quad [\lambda] \mapsto [\lambda \circ \mathrm{Tr}_{\pi_2} \circ \alpha^* \circ \pi_1^*]$$

The reason for the choice $k=2$ is that it is well-known (see[4] [7] §3.3) that in this case there is an isomorphism $\omega : \mathcal{S}_2(\Gamma) \xrightarrow{\sim} \Omega^1(X(\Gamma))$ of $\mathbf{C}$-vector spaces, given essentially by $f \mapsto \omega(f)$ where $\omega(f)$ pulls back to $f(z)\mathrm{d}z$ on $\mathfrak{h}$. This gives rise to the isomorphism $\omega^\vee : \Omega^1(X(\Gamma))^\vee \xrightarrow{\sim} \mathcal{S}_2(\Gamma)^\vee$. Under this isomorphism, the Jacobian $\mathrm{Jac}(X)$ of the modular curve $X = X(\Gamma)$ is identified with the quotient

$$\mathrm{Jac}(X) \cong \mathcal{S}_2(\Gamma)^\vee / \omega^\vee H_1(X;\mathbf{Z}).$$

But now the crux of the argument is that it is easy to see that the following massive diagram commutes:

[3] The choice of notation will be explained momentarily.

[4] Beware of the slightly different notation. Diamond and Shurman use Ω to denote what we call $\mathcal{M}$, and Ω_{hol} to denote what we call Ω.

$$\begin{tikzcd}
\mathcal{S}_2(\Gamma_1) \arrow[r, hook, "{[\Gamma_1 \,\mathrm{id}_2\, \Gamma_3']_2}"] \arrow[d, "\wr"] \arrow[rrr, bend left, "{[\Gamma_1\alpha\Gamma_2]_2}"] & \mathcal{S}_2(\Gamma_3') \arrow[r, "{[\Gamma_3'\alpha\Gamma_3]_2}", "\sim"'] \arrow[d, "\wr"] & \mathcal{S}_2(\Gamma_3) \arrow[r, two heads, "{[\Gamma_3 \,\mathrm{id}_2\, \Gamma_2]_2}"] \arrow[d, "\wr"] & \mathcal{S}_2(\Gamma_2) \arrow[d, "\wr"] \\
\Omega^1(X_1) \arrow[r, "\pi_1^*"] & \Omega^1(X_3') \arrow[r, "\alpha^*"] & \Omega^1(X_3) \arrow[r, "\mathrm{Tr}_{\pi_2}"] & \Omega^1(X_2) \\
X_1 & X_3' \arrow[l, "\pi_1"'] & X_3 \arrow[l, "\alpha"', "\sim"] \arrow[r, "\pi_2"] & X_2
\end{tikzcd}$$

This is essentially because the differential forms transform correctly since for $k=2$, we have $k/2 = k-1 = 1$. Since the below map dualizes to descend to Jacobians, so does the above map (this also explains the notation). Henceforth, we implicitly identify $\mathrm{Jac}(X)$ with a quotient of $\mathcal{S}_2(\Gamma)^\vee$. In this language, we have thus shown:

Proposition 2.3.1. Let $\Gamma_1, \Gamma_2 \leq \mathrm{SL}_2(\mathbf{Z})$ be congruence subgroups, and let $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$. Let $X_i = X(\Gamma_i)$ for $i = 1,2$ be the corresponding modular curves. Then the double coset operator $[\Gamma_1\alpha\Gamma_2]_2$ acts on Jacobians of X_i from the right, i.e. we have an induced linear map

$$[\Gamma_1\alpha\Gamma_2]_2^\vee : \mathrm{Jac}(X_2) \to \mathrm{Jac}(X_1),$$

where $\mathrm{Jac}(X_i) = \mathcal{S}_2(\Gamma_i)^\vee/\omega_i^\vee H_1(X_i;\mathbf{Z})$ for $i = 1,2$, given simply by

$$[\psi] \mapsto [\psi \circ [\Gamma_1\alpha\Gamma_2]_2] \text{ for } \psi \in \mathcal{S}_2(\Gamma_2)^\vee.$$

Now, since the Hecke and diamond operators are (sums of) double coset operators, we have shown:

Proposition 2.3.2. For $k = 2$ and any $n, N \geq 1$, the Hecke operators $T = T_n$ and the diamond operators $T = \langle n \rangle$ act by composition on the Jacobian $J_1(N) := \mathrm{Jac}(X_1(N)))$ associated to $\Gamma_1(N)$,

$$T : J_1(N) \to J_1(N), \quad [\psi] \mapsto [\psi \circ T] \text{ for } \psi \in \mathcal{S}_2(\Gamma_1(N))^\vee.$$

This is at the heart of what we want to show.

2.4 Hecke Algebra and Algebraic Eigenvalues

We are now in a position to prove the unexpected results we wanted.

Theorem 2.4.1. Let $f \in \mathcal{S}_2(\Gamma_1(N))$ be a normalized eigenform. Then the eigenvalues (and hence the Fourier coefficients) $a_n(f)$ are algebraic integers.

This theorem is surprising because *a priori*, there is no reason whatsoever that the coefficients should be algebraic over $\mathbf{Q}$, let alone algebraic integers.

Proof. The Hecke operators $T = T_n$ and $T = \langle n \rangle$ act on the dual space of weight-2 cusp forms by right-composition

$$T : \mathcal{S}_2(\Gamma_1(N))^\vee \to \mathcal{S}_2(\Gamma_1(N))^\vee, \quad \psi \mapsto \psi \circ T.$$

By the previous proposition, the result descends to the quotient $J_1(N)$, so that it acts by endomorphisms on the kernel $H_1 := \omega^\vee H_1(X_1(N);\mathbf{Z})$, a finitely generated abelian group. In particular, the characteristic polynomial f_T of T acting on H_1 is monic and has integral coefficients. By the Cayley-Hamilton Theorem, $f_T(T) \equiv 0$ on H_1. Since T is $\mathbf{C}$-linear and $H_1 \otimes_{\mathbf{Z}} \mathbf{R} = \mathcal{S}_2(\Gamma_1(N))^\vee$, we have $f_T(T) \equiv 0$ on $\mathcal{S}_2(\Gamma_1(N))^\vee$, and so $f_T(T) \equiv 0$ on $\mathcal{S}_2(\Gamma_1(N))$. Therefore, the minimal polynomial of T on $\mathcal{S}_2(\Gamma_1(N))$ divides f_T, and its roots, the eigenvalues, satisfy f_T, making them algebraic integers. ∎

Theorem 2.4.2. The Hecke algebra $\mathbf{T}_{\mathbf{Z}}$ for weight-2 cusp forms for any level $N \geq 1$, when considered as a subalgebra of $\operatorname{End} \mathcal{S}_2(\Gamma_1(N))$, is finitely generated.

Proof. As before, the Hecke and diamond operators, and hence the Hecke algebra, acts on the dual space $\mathcal{S}_2(\Gamma_1(N))^\vee$ by right composition, and this action descends to the quotient. This means that $\mathbf{T}_{\mathbf{Z}}$ can be viewed as a subring of endomorphisms of the free $\mathbf{Z}$-module H_1. But the ring of endomorphisms of a free $\mathbf{Z}$-module of rank g is a free $\mathbf{Z}$-module of rank g^2, and so any subring of it is a free $\mathbf{Z}$-module of finite rank; in particular, it is finitely generated. This means that $\mathbf{T}_{\mathbf{Z}}$, when considered a subalgebra of $\operatorname{End} H_1$, is finitely generated. But now as before, we use $H_1 \otimes_{\mathbf{Z}} \mathbf{R} = \mathcal{S}_2(\Gamma_1(N))^\vee$ to conclude that it is also finitely generated when considered as a subalgebra of $\operatorname{End} \mathcal{S}_2(\Gamma_1(N))$. ■

Theorem 2.4.3. Let $f \in \mathcal{S}_2(\Gamma_1(N)$ be a normalized eigenform, and write $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$. Then the subfield $K_f := \mathbf{Q}(\{a_n\})$ of $\mathbf{C}$ generated by the Fourier coefficients of f is a number field, i.e. a finite-degree extension of $\mathbf{Q}$.

Proof. For a fixed normalized eigenform f, consider the eigenvalue homomorphism

$$\lambda_f : \mathbf{T}_{\mathbf{Z}} \to \mathbf{C} \text{ characterized by } Tf = \lambda_f(T) f.$$

By Proposition 2.2.9, its image is $\mathbf{Z}[\{a_n\}]$. Since $\mathbf{T}_{\mathbf{Z}}$ is a finitely generated $\mathbf{Z}$-module, so is $\mathbf{Z}[\{a_n\}]$. By Theorem 2.4.1, the a_n are algebraic integers, so that the field of fractions of the image is a finite degree extension of $\mathbf{Q}$. ■

Definition 2.4.4. Let $f \in \mathcal{S}_2(\Gamma_1(N))$ be a normalized eigenform, and write its q-expansion $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$. Then the number field $K_f = \mathbf{Q}(\{a_n\})$ generated by the Fourier coefficients of f is called the *number field* of f.

The significance of this theorem is that now we can use different embeddings $\sigma : K_f \hookrightarrow \mathbf{C}$ to produce more eigenforms. More precisely, given an embedding $\sigma : K_f \hookrightarrow \mathbf{C}$, we can conjugate f given by $f(\tau) = \sum_{n \geq 1} a_n q^n$ to produce f^σ given by

$$f^\sigma(\tau) = \sum_{n=1}^{\infty} a_n^\sigma q^n.$$

Then we have:

Theorem 2.4.5. Let $f \in \mathcal{S}_2(\Gamma_1(N))$ be a normalized eigenform. For any $\sigma : K_f \hookrightarrow \mathbf{C}$, the conjugated f^σ is also a normalized eigenform. If f is a newform, then so is f^σ.

This theorem makes sense because there is a sense of symmetry between the different forms in $\{f^\sigma : \sigma \in \operatorname{Gal}(K_f/\mathbf{Q})\}$. We'll exploit this symmetry in the next section. Using this theorem, and standard facts from Galois theory, it can then be shown that:

Corollary 2.4.6. The space $\mathcal{S}_2(\Gamma_1(N))$ has a basis of forms with integer coefficients.

The idea is to symmetrize to show that there is a basis of forms with $\operatorname{Gal}(K_f/\mathbf{Q})$-invariant coefficients. This would mean by Galois theory that the coefficients are in $\mathbf{Q}$, and then Theorem 2.4.1 implies that they are integers. For more precise statements and proofs of the above results, see [7] §6.5.

2.5 Abelian Varieties and Decomposition of Modular Jacobians

The plan for this last section is to use the theory developed above to talk briefly about abelian varieties associated to newforms, and the decomposition of modular Jacobians into a direct sum of abelian varieties associated to classes of newforms. This section is based on [7] §6.6.

Let $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_1(M_f))$ be a newform at some level[5] M_f, and therefore an eigenform for the Hecke algebra $\mathbf{T_Z}$. Recall that we have the eigenvalue homomorphism:

$$\lambda_f : \mathbf{T_Z} \to \mathbf{C} \text{ characterized by } Tf = \lambda_f(T)f.$$

Denote the kernel $\ker \lambda_f$ by I_f, i.e.

$$I_f := \ker \lambda_f = \{T \in \mathbf{T_Z} : Tf = 0\}.$$

The image of this map is the $\mathbf{Z}$-module $\mathbf{Z}[\{a_n\}]$, and by the First Isomorphism Theorem, the map $T \mapsto \lambda_f(T)$ induces a $\mathbf{Z}$-module isomorphism $\mathbf{T_Z}/I_f \xrightarrow{\sim} \mathbf{Z}[\{a_n(f)\}]$, where the latter is an order in $\mathcal{O}_{K_f}$ and consequently has rank $[K_f : \mathbf{Q}]$ as a $\mathbf{Z}$-module. Since $\mathbf{T_Z}$ acts on the Jacobian $J_1(M_f)$, the subgroup $I_f J_1(M_f)$ of $J_1(M_f)$ and the following definition make sense.

Definition 2.5.1. The *abelian variety* associated to f is defined to be the quotient:

$$A_f := J_1(M_f)/I_f J_1(M_f).$$

Now $\mathbf{T_Z}/I_f$ acts on A_f and hence so does its isomorphic image $\mathbf{Z}[\{a_n(f)\}]$. Since by Proposition 2.2.9 we have $\lambda_f(T_n) = a_n(f)$, the following diagram commutes:

$$\begin{tikzcd} J_1(M_f) \arrow[r, "T_n"] \arrow[d, two heads] & J_1(M_f) \arrow[d, two heads] \\ A_f \arrow[r, "a_n(f)"] & A_f. \end{tikzcd}$$

(Note that the bottom row in general is not multiplication by $a_n(f)$, but rather T_n acting on A_f, i.e. for $[\psi] \in A_f, \sigma : K_f \hookrightarrow \mathbf{C}$ we have $(a_n(f)[\psi])(f^\sigma) = a_n(f)^\sigma \psi(f^\sigma)$. However, if $a_n(f) \in \mathbf{Z}[\{a_n\}]$ is in the isomorphic image $\mathbf{Z} \subseteq \mathbf{Z}[\{a_n\}]$ of $\mathbf{Z} + I_f \subseteq \mathbf{T_Z}/I_f$, then it does act by multiplication.)

We deduce now the complex-analytic structure of abelian varieties associated to newforms. For that, we first define an equivalence relation on newforms by

$$f \sim f' \Leftrightarrow \exists \sigma \in \mathrm{Aut}(\mathbf{C}) : f^\sigma = f',$$

and let $[f] = \{f^\sigma : \sigma \in \mathrm{Aut}(\mathbf{C})\}$ be the equivalence class of f; then by Theorem 2.4.5, $|[f]| = [K_f : \mathbf{Q}]$, and each f^σ is a newform at level M_f. Consider the subspace

$$V_f := \mathbf{C}\langle [f] \rangle \subseteq \mathcal{S}_2(\Gamma_1(M_f)).$$

By Proposition 2.2.12, this space has dimension $[K_f : \mathbf{Q}]$. Restricting the discrete subgroup $\omega^\vee H_1(X_1(M_f);\mathbf{Z})$ of $\mathcal{S}_2(\Gamma_1(M_f))^\vee$ to V_f gives the subgroup

$$\Lambda_f := \omega^\vee H_1(X_1(M_f);\mathbf{Z})|_{V_f} \subseteq V_f^\vee.$$

Restricting to V_f gives a well-defined homomorphism

$$J_1(M_f) \to V_f^\vee/\Lambda_f, \quad [\psi] \mapsto \psi|_{V_f} + \Lambda_f \text{ for } \psi \in \mathcal{S}_2(\Gamma_1(M_f))^\vee.$$

The claim is that this induces an isomorphism on the corresponding abelian variety.

[5] Note that f uniquely determines M_f, although we don't show this. Essentially, M_f is simply the smallest integer M such that $f \in \mathcal{S}_2(\Gamma_1(M))$. See [7] Chapter 5 for more details.

Proposition 2.5.2. Let $f \in \mathcal{S}_2^{\text{new}}(\Gamma_1(M_f))$ be a newform with number field K_f. Then restricting to V_f induces an isomorphism

$$A_f \xrightarrow{\sim} V_f^\vee/\Lambda_f, \quad [\psi] + I_f J_1(M_f) \mapsto \psi|_{V_f} + \Lambda_f \text{ for } \psi \in \mathcal{S}_2(\Gamma_1(M_f))^\vee,$$

and the right side is a $[K_f : \mathbf{Q}]$-dimensional complex torus. In particular, A_f is a $[K_f : \mathbf{Q}]$-dimensional complex torus.

Proof Sketch. For the sake of brevity, let $\mathcal{S}_2 = \mathcal{S}_2(\Gamma_1(M_f))$ and $H_1 = \omega^\vee H_1(X_1(M_f);\mathbf{Z})$. Using a bunch of named isomorphism theorems,

$$A_f = J_1(M_f)/I_f J_1(M_f) = (\mathcal{S}_2^\vee/H_1)/I_f(\mathcal{S}_2^\vee/H_1) \cong \mathcal{S}_2^\vee/(I_f\mathcal{S}_2^\vee + H_1) \cong (\mathcal{S}_2^\vee/I_f\mathcal{S}_2^\vee)/\overline{H_1},$$

where $\overline{H_1}$ is the image of H_1 in $\mathcal{S}_2^\vee/I_f\mathcal{S}_2^\vee$. Let $\mathcal{S}_2[I_f] := \{g \in \mathcal{S}_2 : \exists T \in I_f : Tg = 0\}$. The restriction map $\mathcal{S}_2^\vee \twoheadrightarrow \mathcal{S}_2[I_f]^\vee$ induces the isomorphism

$$\mathcal{S}_2^\vee/I_f\mathcal{S}_2^\vee \xrightarrow{\sim} \mathcal{S}_2[I_f]^\vee, \quad [\psi] + I_f\mathcal{S}_2^\vee \mapsto \psi|_{\mathcal{S}_2[I_f]},$$

so that

$$A_f \xrightarrow{\sim} \mathcal{S}_2[I_f]^\vee/H_1|_{\mathcal{S}_2[I_f]}, \quad [\psi] + I_f J_1(M_f) \mapsto \psi|_{\mathcal{S}_2[I_f]} + H_1|_{\mathcal{S}_2[I_f]}.$$

It suffices to show that $\mathcal{S}_2[I_f] = V_f$ and that $\Lambda_f = H_1|_{V_f} \subseteq V_f^\vee$ is a full rank lattice. Clearly, $V_f \subseteq \mathcal{S}_2[I_f]$. To show the other direction, first show using Proposition 2.2.9 that the pairing

$$\mathbf{T_C} \times \mathcal{S}_2 \to \mathbf{C}, \quad (T,g) \mapsto a_1(Tg)$$

is a perfect pairing of $\mathbf{T_Z}$-modules. This induces the isomorphism $\mathcal{S}_2^\vee \xrightarrow{\sim} \mathbf{T_C}$ so that

$$\dim_{\mathbf{C}} \mathcal{S}_2[I_f] = \dim_{\mathbf{C}} \mathcal{S}_2[I_f]^\vee = \dim_{\mathbf{C}} \mathcal{S}_2^\vee/I_f\mathcal{S}_2^\vee = \dim_{\mathbf{C}} \mathbf{T_C}/I_f\mathbf{T_C}.$$

But the natural surjection $\mathbf{T_Z} \otimes_{\mathbf{Z}} \mathbf{C} \twoheadrightarrow \mathbf{T_C}, \sum U_i \otimes z_i \mapsto \sum z_i U_i$ descends to a surjection $(\mathbf{T_Z}/I_f) \otimes_{\mathbf{Z}} \mathbf{C} \xrightarrow{\sim} (\mathbf{T_Z} \otimes_{\mathbf{Z}} \mathbf{C})/(I_f \otimes_{\mathbf{Z}} \mathbf{C}) \twoheadrightarrow \mathbf{T_C}/I_f\mathbf{T_C}$, so that

$$\dim_{\mathbf{C}} \mathcal{S}_2[I_f] \leq \dim_{\mathbf{C}}(\mathbf{T_Z}/I_f) \otimes_{\mathbf{Z}} \mathbf{C} = \operatorname{rank}(\mathbf{T_Z}/I_f) = [K_f : \mathbf{Q}] = \dim_{\mathbf{C}} V_f.$$

Finally, it suffices to show that $\Lambda_f \otimes_{\mathbf{Z}} \mathbf{R} \supseteq V_f^\vee$ and $\operatorname{rank}(\Lambda_f) \leq \dim_{\mathbf{R}}(V_f^\vee)$. Since $V_f \hookrightarrow \mathcal{S}_2$, the restriction $\pi : \mathcal{S}_2^\vee \twoheadrightarrow V_f^\vee$ surjects; but then since $H_1 \otimes_{\mathbf{Z}} \mathbf{R} = \mathcal{S}_2^\vee$, we have $\Lambda_f \otimes_{\mathbf{Z}} \mathbf{R} = \pi(H_1) \otimes_{\mathbf{Z}} \mathbf{R} \supseteq V_f^\vee$. On the other hand,

$$\dim_{\mathbf{R}} V_f^\vee = \dim_{\mathbf{R}}(\mathcal{S}_2^\vee/I_f\mathcal{S}_2^\vee) = \dim_{\mathbf{R}}(H_1 \otimes_{\mathbf{Z}} \mathbf{R})/(I_f H_1 \otimes_{\mathbf{Z}} \mathbf{R}) = \operatorname{rank}(H_1/I_f H_1).$$

Now $\Lambda_f = \pi_(H_1) \cong H_1/(H_1 \cap \ker \pi)$ and $I_f H_1 \subseteq \ker \pi$ show us that there is a surjection $H_1/I_f H_1 \twoheadrightarrow \Lambda_f$, making $\operatorname{rank}(\Lambda_f) \leq \operatorname{rank}(H_1/I_f H_1) = \dim_{\mathbf{R}} V_f^\vee$. ■

The useful equivalence relation between complex tori turns out to be weaker than a complex Lie group isomorphism–it is an *isogeny*.

Definition 2.5.3. An *isogeny* of complex tori is a surjective holomorphic homomorphism with finite kernel.

It is easy to prove that like in the elliptic curve $g = 1$ case, isogenies are equivalence relations. The key theorem in this direction is then:

Theorem 2.5.4. The Jacobian $J_1(N)$ associated to $\Gamma_1(N)$ is isogenous to a direct sum of abelian varieties associated to equivalence classes of newforms,

$$J_1(N) \to \bigoplus_{[f]} A_f^{\sigma_0(N/M_f)},$$

where the sum is over equivalence classes over newforms $f \in \mathcal{S}_2^{\text{new}}(\Gamma_1(M_f))$ for levels $M_f|N$, and $\sigma_0(N/M_f)$ is the number of positive divisors of N/M_f.

For a proof, see [7] Theorem 6.6.6.

3 Summary

Associated to any compact Riemann surface X of genus g, we have a g-dimensional complex torus

$$\mathrm{Jac}(X) := \frac{\Omega^1(X)^\vee}{\int_{-} H_1(X;\mathbf{Z})}.$$

We have the Abel-Jacobi map $A : X \to \mathrm{Jac}(X)$, which is injective for $g \geq 1$. This extends to a canonical map $A_0 : \mathrm{Div}^0(X) \to \mathrm{Jac}(X)$, which then by the Abel-Jacobi Theorem descends to an isomorphism

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X) \text{ given by } \left[\sum_p n_p(p)\right] \mapsto \left[\sum_p n_p \int^p\right],$$

where the integrals are all taken from the same point. Associated to a nonconstant holomorphic map $F : X \to Y$ of compact Riemann surfaces, we then define the forward maps F_J, F_P, F_D and the backward maps F^J, F^P, F^D of Jacobians, Picard groups, and divisor groups respectively, which are compatible under the Abel-Jacobi isomorphism and the natural quotient map.

Given congruence subgroups $\Gamma_1, \Gamma_2 \leq \mathrm{SL}_2(\mathbf{Z})$, $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, and $k \in \mathbf{Z}$, we have the double coset operators

$$[\Gamma_1 \alpha \Gamma_2]_k : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2),$$

which preserve cusp forms. Special cases of these operators include the Hecke and diamond operators. By interpreting the double coset operators geometrically on the modular curves $X(\Gamma_1)$ and $X(\Gamma_2)$ and using the above forward and backward maps, it is then shown that these double coset operators act on the Jacobians by composition from the right. This means in particular that Hecke and diamond operators preserve the finitely generated abelian groups $H_1(X(\Gamma_i);\mathbf{Z})$, and this proves many interesting results including the fact that by adjoining the Fourier coefficients of a normalized eigenform $f \in \mathcal{S}_2(\Gamma_1(N))$ to $\mathbf{Q}$, we get a number field K_f. The Galois theory of this number field then allows us to generate new normalized eigenforms by conjugation by elements of $\mathrm{Gal}(K_f/\mathbf{Q})$, and allows us to prove that $\mathcal{S}_2(\Gamma_1(N))$ has a basis of forms with integer coefficients.

Associated to each newform $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_1(M_f))$ is the abelian variety A_f, which is a $[K_f : \mathbf{Q}]$-dimensional complex torus. The main theorem in this direction is that the Jacobian $J_1(N) := \mathrm{Jac}(X_1(N)))$ then decomposes as a direct sum of abelian varieties associated to equivalence classes of newforms in $\mathcal{S}_2(\Gamma_1(N))$. Finally, we mention that the famed Modularity Theorem of Breuil, Conrad, Diamond, Taylor and Wiles can be stated purely in terms of abelian varieties.

Theorem 3.0.1 (Modularity Theorem, Version $A_{\mathbf{C}}$)**.** Let $E = \mathbf{C}/\Lambda$ be a complex elliptic curve with j-invariant $j(E) \in \mathbf{Q}$. Then for some positive integer N and some newform $f \in \mathcal{S}_2(\Gamma_0(N)) \subseteq \mathcal{S}_2(\Gamma_1(N))$, there exists a surjective holomorphic homomorphism of complex tori

$$A_f \to E.$$

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