Representation Theory Review Notes

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1 Preface

These are the notes on Representation Theory based on the first few chapters of Fulton and Harris's *Representation Theory: A First Course*. Most of this material is expansion of the material from Fulton-Harris, with notation borrowed from Serre for clarity. In particular, I include no examples that Fulton and Harris do in their book.

2 Introduction

Definition 1. Let G be a group and k be a field. A *representation of* G over k is a pair (V, ρ), where V is a vector space over k and $\rho : G \to GL(V)$ a homomorphism from G to the group of invertible linear maps (i.e. automorphisms) of V.

Remark 1. As opposed to the more customary $\rho(g)$ used to denote ρ applied to a $g \in G$, we use the notation ρ_g to emphasise that ρ_g is not an element of V, but an invertible linear map $\rho_g : V \to V$. The condition of ρ being a homomorphism then reads $\rho_{g\cdot h} = \rho_g \circ \rho_h$, so that e.g. $\rho_e = \mathrm{id}_V$.

Remark 2. Some authors define a k-representation of a group G as a vector space V over k with a map

$$G \times V \to V, \quad (g, v) \mapsto gv$$

satisfying the axioms of

(1) linearity: for each $g \in G$, the map $v \mapsto gv$ as a map $V \to V$ is a k-linear map, i.e. $g(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g v_1 + \lambda_2 g v_2$ for all $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in k$,

(2) and group action: $(gh)v = g(hv)$ and $ev = v$ for all $g, h \in G$ and $v \in V$.

The second axiom ensures that $g^{-1}(gv) = ev = v$ for all $g \in G$, so that each $g \in G$ gives rise to an invertible linear map $g : V \to V$ given by $v \mapsto gv$ with inverse $g^{-1} : V \to V$ given by $v \mapsto g^{-1}v$. This tells us that this definition is equivalent to the one above, with the translation being that if (V, ρ) is as above, then we get $G \times V \to V$ by $(g, v) \mapsto \rho_g(v)$. Therefore, you should think of applying $\rho_g : V \to V$ to a vector as "left-multiplying it by g", although this doesn't make literal sense since you can't multiply an element of a group G with a vector in a vector space V.

Remark 3. Mathematicians are lazy, as usual, and often drop either V or ρ from the pair (V, ρ), calling the remainder a representation. For example, Fulton and Harris call V a representation with the ρ being implicit (c.f. Remark 2), while Serre calls ρ itself a representation. When learning this material for the first time, it's helpful to remember that a representation is actually a *pair* of things: a k-vector space V, and an action ρ of G on V by linear maps. This is analogous to keeping in mind, e.g., that a group more properly is a pair $(G, \cdot)$, where G is a set and $\cdot$ a law of composition satisfying certain axioms; as you get "older", you'll get more comfortable with the language that either text uses.

Remark 4. In this course, we are going to be primarily interested in the case when $k = \mathbb{C}$ and G is a finite group, although other examples were discussed in class (e.g. with $G = SL_2 \mathbb{Z}$ or a braid group, and $k = \mathbb{R}$).

Example 1. Observe that if 0 is the 0-vector space, then $GL(0) \cong \{e\}$. If G is any group, then there is a unique homomorphism $\rho : G \to GL(0)$, so that for any group G we have the representation $(0, \rho)$. This is called the *zero representation*, and is usually denoted $(0, 0)$. (This is the additive identity of the representation ring $R_k(G)$, discussed below.)

Example 2. If $G = \{e\}$ is the trivial group, then there is only one possible homomorphism $\rho : G \to GL(V)$ for any vector space V. Therefore, a representation of the trivial group is the same thing as a vector space V, with e acting by $ev = v$ for all $v \in V$.

Example 3. If $G = \mathbb{Z} \cong F_1 = \langle a \rangle$ is the free group on one generator, then a homomorphism $\rho : G \to GL(V)$ is the same thing as picking a single element $\rho_a =: \varphi \in GL(V)$. Therefore, a representation of $\mathbb{Z}$ is the same thing as a vector space V with a choice of a distinguished automorphism φ of V.

Example 4. More generally, if $G = F_n = \langle a_1, \dots, a_n \rangle$ is the free group on n-generators, then a representation of G is the same thing as a vector space V with a choice of n distinguished (but not necessarily distinct) automorphisms $\varphi_1, \dots, \varphi_n$ of V. These are not required to satisfy any constraints at all.

Example 5. If $G = \mathbb{Z}/2 = \langle a | a^2 = e \rangle$, then a homomorphism $\rho : G \to GL(V)$ is the same thing as the choice of an automorphism φ of V such that $\varphi^2 = \mathrm{id}_V$. (More generally, a homomorphism $\rho : \mathbb{Z}/2 \to H$ from $\mathbb{Z}/2$ to any group H is the same thing as a choice of an element h of H such that $h^2 = e$. The possibility $h = e$ is allowed. Make sure you understand this!) Therefore, a representation of $\mathbb{Z}/2$ is the same thing as a vector space V with a choice of a distinguished automorphism φ of V satisfying $\varphi^2 = \mathrm{id}_V$.

Example 6. Generalizing the previous example, if $G = \mathbb{Z}/n = \langle a | a^n = e \rangle$ for some $n \geq 1$, then a representation of G is the same thing as a vector space V with a choice of a distinguished automorphism φ of V satisfying $\varphi^n = \mathrm{id}_V$.

Example 7. If $G = \mathbb{Z} \times \mathbb{Z} = \langle a, b | aba^{-1}b^{-1} = e \rangle$, then a representation of G is the same thing as a vector space V with a choice of distinguished automorphisms φ, ψ of V that commute with each other, i.e. that satisfy $\varphi \circ \psi = \psi \circ \varphi$.

Remark 5. The preceding examples show that a representation of a finitely generated group G is the same thing as a vector space V with a choice of automorphism for each generator of G satisfying the same relations as the correspond generators of G. The idea of a representation of an arbitrary group is a vast generalization of this idea.

Remark 6. In the example of $G = \mathbb{Z}/n$ above, we could have very well taken $\varphi = \mathrm{id}_V$, which works because $\mathrm{id}_V^n = \mathrm{id}_V$. In general, there may be "collapsing". For example, for any group G, we may take any vector space V and take $\rho : G \to GL(V)$ to be the trivial homomorphism, so that $\rho_g = \mathrm{id}_V$ for all

$g \in G$; this is trivial and not very interesting of a representation. If indeed there is no collapsing, or equivalently if the homomorphism $\rho : G \to GL(V)$ is injective, then we call the representation (V, ρ) a *faithful representation*. In general, this is a stringent requirement on a representation.

Having defined representations, let's now turn to homomorphisms between them, and standard constructions.

Definition 2. Suppose (V, ρ) and (W, π) are k-representation of a group G. A *homomorphism of representations* $\phi : (V, \rho) \to (W, \pi)$ between (V, ρ) and (W, π) is a k-linear map $\phi : V \to W$ on the underlying vector spaces that respects their representation structure, i.e. such that for every $g \in G$, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \pi_g} \\ V & \xrightarrow{\phi} & W \end{array}$$

of linear maps of k-vector spaces commutes, i.e. for every $g \in G$ we have $\phi \circ \rho_g = \pi_g \circ \phi$.

Remark 7. When we think of a representation action ρ_g as "left-multiplication", then the above statement becomes the more palatable statement that a k-linear map ϕ of the underlying vector spaces V and W of representations (V, ρ) and (W, π) of G is a homomorphism of representations iff for every $g \in G$ and $v \in V$ we have $\phi(gv) = g\phi(v)$. This formulation hides the fact that g is acting differently in general on the two sides of the equation–indeed it is acting on elements of different vector spaces on the two sides.

Remark 8. A homomorphism of representations of a group G is also called a G-*linear map*, or a G-*equivariant map*, or an *intertwiner*, or a G-*module homomorphism* (for good reason!).

Example 8. The identity map $\mathrm{id}_V : (V, \rho) \to (V, \rho)$ from any representation to itself is a homomorphism of representations. More generally, any scalar multiple of the identity $\lambda \, \mathrm{id}_V$ for $\lambda \in k$ is a homomorphism of representations.

Observe that if ϕ and ψ a homomorphism of representations $(V, \rho) \to (W, \pi)$, then so are $\lambda\phi + \mu\psi$ for every choice of $\lambda, \mu \in k$ (make sure you understand what this means!). This tells us that the set of all homomorphisms of representations from (V, ρ) to (W, π) is a k-subspace of $\mathrm{Hom}(V, W)$. This subspace is denoted by $\mathrm{Hom}_G(V, W)$. Strictly speaking, this depends on ρ and π, so we should be writing something like $\mathrm{Hom}_G((V, \rho), (W, \pi))$, but see Remark 3.

3 Constructing New Representations

Let's look at some ways of constructing new representations out of old ones.

Definition 3. Suppose (V, ρ) is a k-representation of a group G. Then a *subrepresentation of* (V, ρ) is a k-subspace $U \subseteq V$ that is invariant under the action of G, i.e. a k-subspace U of V such that for every $g \in G$ we have that $\rho_g U \subseteq U$.

Remark 9. A subrepresentation is usually denoted by $(U, \rho|_U) \subseteq (V, \rho)$, although strictly speaking $\rho|_U$ doesn't make sense because ρ doesn't act on U. What we mean by this notation is that $\rho|_U : G \to GL(U)$ is the map given by taking an element $g \in G$ to the restriction $\rho_g|_U$ of the linear map $\rho_g : V \to V$ to the subspace U. (Why is this restriction invertible?)

Remark 10. A subrepresentation can also be though of as the image of an injective homomorphism of representations. (Why?)

Lemma 1. *Suppose* $\phi : (V, \rho) \to (W, \pi)$ *is a homomorphism of representations. Then the kernel* $(\ker \phi, \rho|_{\ker \phi}) \subseteq (V, \rho)$ *is a subrepresentation, and similarly the image* $(\mathrm{im}\, \phi, \pi|_{\mathrm{im}\, \phi}) \subseteq (W, \pi)$.

Proof. The kernel $\ker\phi \subseteq V$ of the underying linear map is a k-subspace, so we want to check that it is G-invariant, i.e. for every $g \in G$ we have $\rho_g \ker\phi \subseteq \ker\phi$. Suppose $v \in \ker\phi$; then we want to check that $\rho_g v \in \ker\phi$, i.e. $\phi(\rho_g v) = 0$. But that is true because

$$\phi(\rho_g v) = \pi_g(\phi(v)) = \pi_g(0) = 0,$$

which uses crucially the fact that ϕ is a homomorphism of representations. Similarly, the image $\operatorname{im}\phi \subseteq W$ is a k-subspace, and it is G-invariant (i.e. invariant under the action of π_g for every $g \in G$) because if $w = \phi(v) \in \operatorname{im}\phi$, then $\pi_g w = \pi_g\phi(v) = \phi(\rho_g v) \in \operatorname{im}\phi$. ■

Example 9. Let (V, ρ) be any k-representation of a group G. Define the *maximal trivial subrepresentation* $(V^G, \rho|_{V^G})$ of (V, ρ) to be the G-invariant subspace

$$V^G := \{v \in V : \forall g \in G, \rho_g v = v\}.$$

This is the largest subrepresentation of V on which G acts trivially.

Definition 4. Suppose (V, ρ) is a k-representation of a group G, and suppose $(U, \rho|_U) \subseteq (V, \rho)$ is a subrepresentation. Then we define the *quotient representation* $(V/U, \overline{\rho})$ as follows: take the underlying vector space to be V/U, and define $\overline{\rho} : G \to GL(V/U)$ by defining for $g \in G$ the map $\overline{\rho}_g : V/U \to V/U$ to be the map $[v] \mapsto [\rho_g v]$. (Why is this well-defined irrespective of the choice of representative $v \in V$ of the coset $[v] \in V/U$? Why is this map invertible?)

Definition 5. A k-representation (V, ρ) of a group G is said to be *irreducible* if it is nonzero and the only subrepresentations of (V, ρ) are the trivial subrepresentation $(0, \rho|_0)$ and all of (V, ρ). In other words, a representation (V, ρ) is irreducible if no nontrivial proper subspace of V is G-invariant, i.e. if it holds that if $0 \subsetneq U \subsetneq V$ is any nontrivial proper subspace of V, then there is some $g \in G$ such that $\rho_g U \not\subseteq U$.

Remark 11. In a sense, you can think of as a subspace of a representation as analagous to a subgroup of a group, with subrepresentations corresponding to normal subgroup. This is because if you quotient out a representation by a mere subspace, you will not in general get another representation, but if you quotient out a representation by a sub*representation*, then you do indeed get a representation, as we have seen above. In this sense, irreducible representations are similar to simple groups. Since the classification of finite groups comes down to classifying finite groups, the classification of finite-dimensional representations of a given finite group should also come down to classifying its irreducible representations. That is indeed the case, at least in the special scenario of finite dimensional $\mathbb{C}$-representations of finite groups.

Lemma 2 (Schur's Lemma). *If (V, ρ) and (W, π) are irreducible representations of G and $\phi : (V, \rho) \to (W, \pi)$ a homomorphism of representations, then*

(1) either ϕ is an isomorphism of representations, or $\phi = 0$. In particular, if (V, ρ) and (W, π) are nonisomorphic representations, then $\phi = 0$.
(2) If indeed ϕ is an isomorphism, $(V, \rho) \cong (W, \pi)$ is finite dimensional, and k is algebraically closed (e.g. if $k = \mathbb{C}$), then implicitly identifying (V, ρ) and (W, π) via ϕ, we have that $\phi : V \to V$ is just $\lambda\,\mathrm{id}_V$ for some $\lambda \in k$.

Proof.

(1) By the previous lemma, the kernel $(\ker\phi, \rho|_{\ker\phi}) \subseteq (V, \rho)$ is a subrepresentation. By definition of irreducibility, either $(\ker\phi, \rho|_{\ker\phi})$ is $(0, \rho|_0)$ or all of (V, ρ). If $(\ker\phi, \rho|_{\ker\phi}) = (V, \rho)$, then $\phi = 0$. Otherwise, $(\ker\phi, \rho|_{\ker\phi}) = (0, \rho|_0)$ and ϕ is injective. In that case, again by the previous lemma, the image $(\operatorname{im}\phi, \pi|_{\operatorname{im}\phi}) \subseteq (W, \pi)$ is a subrepresentation. This is nonzero (because V is nonzero and ϕ is injective), so that again by irreducibility of (W, π) we must have that $(\operatorname{im}\phi, \pi|_{\operatorname{im}\phi}) = (W, \pi)$, and ϕ is surjective and hence an isomorphism.
(2) Suppose now that k is algebraically closed, and $\phi : (V, \rho) \to (V, \rho)$ is a homomorphism of representations, with V finite dimensional. Then ϕ has an eigenvector, i.e. a $0 \neq v \in V$ with eigenvalue $\lambda \in k$ satisfying $\phi v = \lambda v$. Now, by Example 8 and the following discussion, the map $\phi - \lambda\,\mathrm{id}_V : V \to V$ is also a homomorphism of representations, but this time with nonzero kernel, since $v \in \ker(\phi - \lambda\,\mathrm{id}_V)$. Since $(\ker(\phi - \lambda\,\mathrm{id}_V), \rho|_{\ker(\phi - \lambda\,\mathrm{id}_V)}) \subseteq (V, \rho)$ is a nontrivial subrepresentation, the irreducibility of (V, ρ) tells us that $\ker(\phi - \lambda\,\mathrm{id}_V) = V$, i.e. that $\phi = \lambda\,\mathrm{id}_V$.

■

Corollary 0.1. *If* (V,ρ) *and* (W,π) *are finite-dimensional irreducible* k*-representations of a group* G *over an algebraically closed field* k*, then*

$$\dim \operatorname{Hom}_G(V,W) = \delta^{(V,\rho)}_{(W,\pi)} = \begin{cases} 0, & (V,\rho) \not\cong (W,\pi), \\ 1, & (V,\rho) \cong (W,\pi). \end{cases}$$

Definition 6. Suppose (V,ρ) and (W,π) are k-representations of a group G. Then we define the *direct sum representation* $(V,\rho) \oplus (W,\pi) := (V \oplus W, \rho \oplus \pi)$ as follows: take the underlying vector space to be the direct sum $V \oplus W$, and take the homomorphism $\rho \oplus \pi : G \to GL(V \oplus W)$ to be given by the map that takes $g \in G$ to the automorphism $\rho_g \oplus \pi_g : V \oplus W \to V \oplus W$ given by $(\rho_g \oplus \pi_g)(v,w) = (\rho_g v, \pi_g w)$. We similarly define the direct sum of any number (not necessarily finite) of representations.

Definition 7. Suppose (V,ρ) and (W,π) are representations of a group G. Then we define the *tensor product representation* $(V,\rho) \otimes (W,\pi) := (V \otimes W, \rho \otimes \pi)$ as follows: take the underlying vector space to be $V \otimes W$, and take the homomorphism $\rho \otimes \pi : G \to GL(V \otimes W)$ to be given by the map that takes $g \in G$ to the automorphism $\rho_g \otimes \pi_g : V \otimes W \to V \otimes W$ given by $(\rho_g \otimes \pi_g)(v \otimes w) = \rho_g v \otimes \pi_g w$ on pure tensors. We similarly define the tensor product of any finite number of representations. (I like to stay away from infinite tensor products.)

Exercise 1. Show that the usual isomorphisms $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$ and $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (U \otimes V)$ are isomorphisms of representations. (This exercise shows that you can turn the set of all finite integer linear combinations of finite dimensional k-representations of a group G into a commutative ring with sum $\oplus$ and product $\otimes$. This is called the representation ring of G, and is written $R_k(G)$ or simply $R(G)$ if k is implicit.)

Definition 8. Suppose (V,ρ) is a representation of a group G. Define the *dual representation* (V^*,ρ^*) as follows: take the underlying vector space to be $V^* := \operatorname{Hom}(V,k)$, and define the homomorphism $\rho^* : G \to GL(V^*)$ according to the following discussion. We have a natural pairing map

$$V^* \times V \to k, \quad (\lambda, v) \mapsto \lambda v.$$

We want to define ρ^* in such a way that preserves this pairing, i.e. such that $\langle \rho^*_g \lambda, \rho_g v\rangle = \langle \lambda, v\rangle$ for all $g \in G, \lambda \in V^*$ and $v \in V$. Since ρ_g is invertible, writing $v = \rho_g^{-1} u$, we get that

$$\langle \rho^*_g \lambda, u\rangle = \langle \rho^*_g \lambda, \rho_g v\rangle = \langle \lambda, v\rangle = \langle \lambda, \rho_g^{-1} u\rangle,$$

so that $(\rho^*_g\lambda)(u) = \lambda(\rho_g^{-1}u) = ({}^t\rho_g^{-1}\lambda)(u)$. This means that we should define $\rho^*_g := {}^t\rho_g^{-1}$.

Definition 9. Suppose (V,ρ) and (W,π) are representations of a group G. Define the representation $(\operatorname{Hom}(V,W), \sigma := \pi \circ - \circ \rho^{-1})$ as follows: take the underlying vector space to be $\operatorname{Hom}(V,W)$, and define a representation structure on it via the k-vector space isomorphism $\operatorname{Hom}(V,W) \cong V^* \otimes W$. In other words, we have commutative diagrams

$$\begin{array}{ccc} V^* \otimes W & \xrightarrow{\sim} & \operatorname{Hom}(V,W) \\ \downarrow{\scriptstyle \rho^*_g \otimes \pi_g} & & \downarrow{\scriptstyle ?} \\ V^* \otimes W & \xrightarrow{\sim} & \operatorname{Hom}(V,W) \end{array} \qquad \begin{array}{ccc} \lambda \otimes w & \longmapsto & v \mapsto \lambda(v) w \\ \downarrow & & \downarrow \\ \lambda \circ \rho_g^{-1} \otimes \pi_g w & \longmapsto & v \mapsto \pi_g(\lambda(\rho_g^{-1} v) w), \end{array}$$

which tells us that we should define the action of g on a map $\operatorname{Hom}(V,W) \ni \phi \mapsto \sum_i \lambda_i \otimes w_i \in V^* \otimes W$ by taking it to $\sum_i \lambda_i \circ \rho_g^{-1} \otimes \pi_g w_i \mapsto \pi_g \circ \phi \circ \rho_g^{-1}$, i.e. in such a way that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \pi_g} \\ V & \xrightarrow{\sigma_g \phi} & W. \end{array}$$

By construction, the isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$ becomes an isomorphism of representations.

Remark 12. It follows immediately from the definition of the action on $\mathrm{Hom}(V, W)$ that the subspace $\mathrm{Hom}_G(V, W) \subseteq \mathrm{Hom}(V, W)$ is actually the maximal trivial subrepresentation $\mathrm{Hom}(V, W)^G$ of $\mathrm{Hom}(V, W)$, i.e.

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G.$$

Remark 13. If (V, ρ) is a representation, then the subspace of symmetric n-tensors $\mathrm{Sym}^n V \subseteq V^{\otimes n}$ is actually a subrepresentation of $(V^{\otimes n}, \rho^{\otimes n})$, and similarly for the alternating n-tensors $\Lambda^n V \subseteq V^{\otimes n}$. These can both also be realized as quotient representations of $V^{\otimes n}$.

Definition 10. If X is any finite set on which the group G acts, then we can form the associated *permutation representation* (V, ρ) by taking $V := k\langle X\rangle$ to be the k-vector space with basis the elements of X, with the action of G on X extending to V by linearity to give ρ. For example, if G is a finite group, then the *(left) regular representation* of G is the permutation representation corresponding to the left action of G on itself by left multiplication.

Remark 14. To avoid confusion, the basis of a permutation representation V corresponding to an action of G on a set X is denoted not by $\{x : x \in X\}$ but $\{e_x : x \in X\}$, as the following example shows.

Example 10. Suppose $G = S_3$ and $X = \{1, 2, 3\}$, with G acting on X in the usual way. (This is the same as the action of D_3 on the vertices of a triangle.) Then the permutation representation looks like $V = k\langle e_1, e_2, e_3\rangle$, with the induced action ρ. For example,

$$\rho_{(123)}(3e_1 + 4e_2 + 5e_3) = 3e_2 + 4e_3 + 5e_1.$$

Definition 11. In general, if $G = S_n$ and $X = \{1, \ldots, n\}$, with G acting on X in the usual way, we get the permutation representation (V, ρ) with $V = k\langle e_1, \ldots, e_n\rangle \cong k^n$. This has a 1-dimensional trivial subrepresentation $(U, \rho|_U)$ given by the span of $\sum_{i=1}^n e_i$ (do you see why?). The quotient representation $(V/U, \overline{\rho})$ is called the *standard representation* of S_n. It can be thought of explicitly as $\left\{\sum_{i=1}^n \lambda_i e_i \in k^n : \sum_{i=1}^n \lambda_i = 0\right\}$, and it has dimension $n - 1$. It is a nontrivial theorem (Fulton and Harris, Proposition 3.12, p. 31) that if $k = \mathbb{C}$, then the standard representations (and more generally all of their wedge powers) are irreducible.

If (V, ρ) is a representation of a group G, then we can ask the following question: for which $g \in G$ is $\rho_g \in \mathrm{Hom}_G(V, V)$? By definition, this happens iff $\forall h \in G : \rho_g \circ \rho_h = \rho_g \circ \rho_g$, i.e. $\forall h \in G : ghg^{-1}h^{-1} \in \ker \rho$. For example, if $g \in Z(G)$ is in the center, then this holds. In particular, if G is abelian, then this holds for every $g \in G$. This already allows us to conclude something about irreducible representations of abelian groups.

Lemma 3. *Any finite-dimensional irreducible representation of a finite abelian group over an algebraically closed field must be 1-dimensional. In particular, if (V, ρ) is a finite dimensional k-representation of a finite abelian group G for an algebraically closed field k (e.g. when $k = \mathbb{C}$), then there is a basis for V such that every ρ_g is diagonal in that basis, i.e. the entire group is simultaneously diagonalizable.*

Proof. Suppose (V, ρ) is a finite-dimensional k-representation of a finite abelian group G for some algebraically closed field k. Then the (finite) family $\{\rho_g : g \in G\}$ of endomorphisms of V commutes with each other, so that by a homework problem, all of these are simultaneously diagonalizable. If V is irreducible and $0 \neq v \in V$ a simultaneous eigenvector for all V, then $0 \subsetneq \langle v\rangle$ is a nontrivial invariant subspace of V (why?), so that irreducibility tells us that $\langle v\rangle = V$, and V is 1-dimensional. ∎

4 Complete Reducibility–Maschke's Theorem

The *fundamental theorem of arithmetic* says that any positive integer can be decomposed uniquely into a product of primes, unique upto the order in which the primes appear. That's a super convenient statement that reduces much of number theory to the study of primes. Does something similar hold for representations?

Q1. Can every k-representation (V, ρ) of a group G be completely decomposed uniquely into a direct sum of irreducible representations?

The answer to this naive question is no for general k, V and G.

Example 11. Complete reducibility may fail if the group G is infinite. For example, take the *shear*, i.e. take $G = (\mathbb{R}, +)$, the vector space $V = \mathbb{R}^2$, and the homomorphism $\rho : \mathbb{R} \to \mathrm{GL}_2 \mathbb{R}$ given by

$$a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

This has the maximal trivial subrepresentation the x-axis, i.e. $V^G = \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x \in \mathbb{R} \right\}$. This is a nontrivial proper subrepresentation, so that (V, ρ) is not irreducible. Nonetheless, it is goemetrically clear that there does not exist a complementary subrepresentation, so that V cannot be written as a direct sum of irreducible representations.

Example 12. Complete reducibility may fail if G is finite, but $\operatorname{char} k$ divides $|G|$. An example of this was on HW 11 Q7. (I'll update this with the actual example if needed later.)

We claim that these are the only problems.

Theorem 2. *Suppose* G *is a finite group and* k *a field such that* $\operatorname{char} k$ *does not divide* $|G|$ *(e.g. if* $\operatorname{char} k = 0$*). If* (V, ρ) *is a* k*-representation of* G *and* $(U, \rho|_U) \subseteq (V, \rho)$ *a subrepresentation, then there is a complementary subrepresentation* $(W, \rho|_W)$ *to* $(U, \rho|_U)$*, i.e. a representation such that the composite* $(U \oplus W, \rho|_U \oplus \rho|_W) \to (V, \rho)$ *is an isomorphism of representations.*

Proof. Pick an arbitrary complementary subspace W_0 of U in V, i.e. a subspace such that $V \cong U \oplus W_0$ as k-vector spaces. Let $\pi_0 : V \to U$ denote projection onto the first factor w.r.t this decomposition. Now define the map

$$\pi : V \to V, \quad \pi(v) := \frac{1}{|G|} \sum_{g \in G} \rho_g(\pi_0(\rho_g^{-1} v)).$$

Since $\operatorname{im} \pi_0 = U$, and $\rho_g U \subseteq U$ for all $g \in G$, we have that $\operatorname{im} \pi \subseteq U$. If $u \in U$, then because for each $g \in G$ we have $\rho_g^{-1} u = \rho_{g^{-1}} u \in U$, it follows that $\pi_0 \rho_g^{-1} u = \rho_g^{-1} u$, and hence $\pi(u) = u$. Therefore, $\pi^2 = \pi$ with $\operatorname{im} \pi = U$ (why?), and so π is also projection onto U. From this, we get a direct sum decomposition $V \cong U \oplus \ker \pi$. It suffices to check that $\ker \pi$ is a subrepresentation of V, and for that, it suffices to prove that $\pi : (V, \rho) \to (V, \rho)$ is a homomorphism of representations. This follows because for any $h \in G$ and $v \in V$ we have that

$$\begin{aligned}
(\pi \circ \rho_h)(v) &= \frac{1}{|G|} \sum_{g \in G} \rho_g(\pi_0(\rho_g^{-1} \rho_h v)) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_h \rho_{h^{-1}g}(\pi_0(\rho_{h^{-1}g}^{-1} v)) \\
&= \rho_h \left(\frac{1}{|G|} \sum_{g \in G} \rho_g(\pi_0(\rho_{g^{-1}} v)) \right) = (\rho_h \circ \pi)(v),
\end{aligned}$$

where the second equality follows from the fact that ρ is a homomorphism, and the second to last equality follows from the linearity of ρ_h and the fact that as g varies over all elements of G, so does $h^{-1}g$. ∎

Remark 15. Make sure you completely understand where in the proof we used both the hypotheses we made.

Remark 16. There is another proof which works only for the case when $k = \mathbb{C}$ that can be found in Fulton and Harris (p. 6). This involves averaging over G as before to get a Hermitian inner product on V invariant under the action of G, and then taking the orthogonal complement of U.

Corollary 2.1 (Complete Reducibility). *If G is a finite group and k a field such that $\operatorname{char} k$ does not divide $|G|$, then every finite-dimensional k-representation (V, ρ) of G can be written as a finite direct sum of irreducible representations.*

Proof. We use strong induction on dimension: if $\dim V = 1$, then V is necessary irreducible (why?). Suppose now that $\dim V = n > 1$, and we have proven the result for all representations of dimension at most $n-1$. If (V, ρ) is irreducible, we are done. Else, (V, ρ) admits a nontrivial proper subrepresentation $(U, \rho|_U)$. By the previous theorem, the hypotheses ensure that there is a complementary subrepresentation $(W, \rho|_W)$ such that $(V, \rho) \cong (U \oplus W, \rho|_U \oplus \rho|_W)$. Since $V = U \oplus W$, we get that $\dim V = \dim U + \dim W$. Since U is both *nontrivial* and *proper*, we get that $1 \leq \dim U, \dim W < \dim V$, so that by the induction hypothesis, these can be written as a direct sum of irreducible representations; then, so can (V, ρ). ■

Now we turn to uniqueness. In this scenario, we want to restrict ourselves to the case of finite dimensional representations of a finite group over an algebraically closed field of characteristic not dividing $|G|$. It is convenient to take $k = \mathbb{C}$.

Theorem 3 (Maschke's Theorem on Complete Reducibility). *If G is a finite group and (V, ρ) a finite-dimensional $\mathbb{C}$-representation of G, then there is a decomposition*

$$(V, \rho) \cong \left(\bigoplus_{i=1}^{n} V_i^{\oplus a_i}, \bigoplus_{i=1}^{n} \rho_i^{\oplus a_i} \right),$$

where the (V_i, ρ_i) are distinct irreducible representations. This decomposition is unique upto the order in which the factors appear, i.e. the integer $n \geq 1$, the irreducibles (V_i, ρ_i), and their multiplicities $a_i \geq 1$ are uniquely determined by (V, ρ).

Proof. The existence of such a decomposition was the content of the previous corollary–of course, we can collect all the isomorphic V_i and put them together by adding to the multiplicity a_i. Next we show uniqueness. Suppose we have two different decompositions $(V, \rho) \cong \left(\bigoplus_{i=1}^n V_i^{\oplus a_i}, \bigoplus_{i=1}^n \rho_i^{\oplus a_i}\right) \cong \left(\bigoplus_{j=1}^m W_j^{\oplus b_j}, \bigoplus_{j=1}^m \pi_j^{\oplus b_j}\right)$. Let $\phi : \left(\bigoplus_{i=1}^n V_i^{\oplus a_i}, \bigoplus_{i=1}^n \rho_i^{\oplus a_i}\right) \to \left(\bigoplus_{j=1}^m W_j^{\oplus b_j}, \bigoplus_{j=1}^m \pi_j^{\oplus b_j}\right)$ denote the isomorphism of representations: we have to show that $n = m$, that each $(V_i, \rho_i) \cong (W_j, \pi_j)$ for a unique j, and that for these i and j we have $a_i = b_j$. It is easily seen that for each i and j, the restriction of ϕ to V_i followed by projection onto W_j, i.e. $\mathrm{pr}_{W_j} \circ \mathrm{res}_{V_i} \phi : V_i \to W_j$ is a homomorphism of representations between the irreducible (V_i, ρ_i) to (W_j, π_j) (why?). Schur's Lemma tells us that this is 0 unless $(V_i, \rho_i) \cong (W_j, \pi_j)$.

(1) First we show that $n = m$. If $n > m$, then because all the (V_i, ρ_i) are pairwise nonisomorphic, and similarly for the (W_j, π_j), there is at least one i such that (V_i, ρ_i) is not isomorphic to any (W_j, ρ_j) for any j. (Why? Use the pigeonhole principle!) Then from the above observation, we must have for this i that $\mathrm{pr}_{W_j} \circ \mathrm{res}_{V_i} \phi = 0$ for every j, so that $\mathrm{res}_{V_i} \phi = 0$ (why does this follow?). This is a contradiction because $(V_i, \rho_i) \not\cong (0, 0)$ (why?), and because ϕ is an isomorphism. (Make sure you understand this point.) Therefore, we have shown that $n > m$ is not possible, and hence $n \leq m$. By symmetry, this means that $m \leq n$, and hence $n = m$.
(2) If for some i, there were no j such that $(V_i, \rho_i) \cong (W_j, \pi_j)$, then ϕ would have a nonzero kernel as before, so that there is at least one j. Since $n = m$ and the (V_i, ρ_i) are pairwise nonisomorphic and similarly for (W_j, π_j), it follows that for each i, there is a *unique* j such that $(V_i, \rho_i) \cong (W_j, \pi_j)$.
(3) For the pair (i, j) in part 2, the restriction $\mathrm{res}_{V_i^{\oplus a_i}} \phi$ can only map into $W_j^{\oplus b_j}$, so that this tells us that the map $\mathrm{res}_{V_i^{\oplus a_i}} \phi : V_i^{\oplus a_i} \to W_j^{\oplus b_j}$ is an isomorphism. By comparing dimensions, we get that $a_i \dim V_i = b_j \dim W_j$. Since $\dim V_i = \dim W_j \neq 0$ by part (2), it follows that $a_i = b_j$.

■

5 Character Theory

We now turn to another powerful tool for analyzing finite-dimensional $\mathbb{C}$-representations of groups.

Definition 12. If (V, ρ) is a finite-dimensional $\mathbb{C}$-representation of a group G, we define the character function $\chi_V : G \to \mathbb{C}$ of (V, ρ) to be given by

$$\chi_V(g) := \operatorname{tr}(\rho_g).$$

Remark 17. The character χ_V is not a homomorphism to $(\mathbb{C}, +)$. (Although if V is one-dimensional, then $\operatorname{im}\chi_V \subseteq \mathbb{C}^*$, and in that case, χ_V is indeed a homomorphism to $(\mathbb{C}^*, \times)$.)

Remark 18. The character χ_V is a class function, i.e. it takes the same value at all elements of a conjugacy class. This is because if $g, h \in G$ are any elements, then

$$\chi_V(hgh^{-1}) = \operatorname{tr}\left(\rho_{hgh^{-1}}\right) = \operatorname{tr}\left(\rho_h \rho_g \rho_h^{-1}\right) = \operatorname{tr}(\rho_g) = \chi_V(g),$$

where the second to last equality follows because $\operatorname{tr}\left(PAP^{-1}\right) = \operatorname{tr}(A)$ for any linear map A and automorphism P.

Remark 19. For the remainder of the section, we will restrict our attention to finite-dimensional $\mathbb{C}$-representations of finite groups. These hypotheses will not be repeated, and will be implicit.

Theorem 4 (Multilinear Algebra of Characters). *Let* (V, ρ) *and* (W, π) *be representations of* G. *Then we have*

(1) $\chi_V(e) = \dim V$,
(2) $\chi_{V \oplus W} = \chi_V + \chi_W$,
(3) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$,
(4) $\chi_{V^*} = \overline{\chi_V}$, *and*
(5) $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$.

Proof. Observe that $\chi_V(e) = \operatorname{tr}(\rho_e) = \operatorname{tr}(\mathrm{id}_V) = \dim V$. For a fixed g, suppose $\{v_1, \dots, v_n\} \subseteq V$ form a basis of eigenvectors for ρ_g with eigenvalues λ_i, and $\{w_1, \dots, w_m\}$ form a basis of eigenvectors for π_g with eigenvalues μ_j, then $\{(v_1, 0), \cdots, (v_n, 0), (0, w_1), \cdots, (0, w_m)\}$ forms a basis of eigenvectors of $V \oplus W$ for $\rho_g \oplus \pi_g$ with eigenvalues $\{\lambda_i\} \cup \{\mu_j\}$, so that

$$\chi_{V \oplus W}(g) = \operatorname{tr}(\rho_g \oplus \pi_g) = \sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \mu_j = \operatorname{tr}(\rho_g) + \operatorname{tr}(\pi_g) = \chi_V(g) + \chi_W(g).$$

Similarly, $\{v_i \otimes w_j\}_{i \in [n], j \in [m]}$ form a basis of eigenvectors of $V \otimes W$ for $\rho_g \otimes \pi_g$ with eigenvalues $\lambda_i \mu_j$, so that

$$\chi_{V \otimes W}(g) = \operatorname{tr}(\rho_g \otimes \pi_g) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \sum \lambda_i \mu_j = \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{j=1}^{m} \mu_j\right) = \operatorname{tr}(\rho_g)\operatorname{tr}(\pi_g) = \chi_V(g)\chi_W(g).$$

If the eigenvalues of ρ_g are λ_i, then the eigenvalues of ρ_g^{-1} are λ_i^{-1}; but now, each λ_i is a root of unity (why?), so that $\lambda_i^{-1} = \overline{\lambda_i}$. This means that (since the trace of a map is the the same as the trace of its dual)

$$\chi_{V^*}(g) = \operatorname{tr}\left(\rho_g^*\right) = \operatorname{tr}\left({}^t\rho_g^{-1}\right) = \operatorname{tr}\left(\rho_g^{-1}\right) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{\operatorname{tr}(\rho_g)} = \overline{\chi_V(g)}.$$

Finally, (5) follows because $(\mathrm{Hom}(V, W), \sigma) \cong (V^* \otimes W, \rho^* \otimes \pi)$ as representations, and isomorphic representations have the same characters (why?). ■

Corollary 4.1. *The map* $\chi : R_{\mathbb{C}}(G) \to \mathbb{C}^G_{cl}$ *is a ring homomorphism.*

Proof. This follows from (2) and (3). ■

Corollary 4.2. *If* (V, ρ) *is a representation s.t.* $\chi_V(g)$ *is nonreal for some* $g \in G$, *then* $(V, \rho) \not\cong (V^*, \rho^*)$.

Proof. This follows from (4), since isomorphic representations have the same character. ■

Remark 20. Using similar arguments as in the above proof, formulae for the characters of $\mathrm{Sym}^k V$ and $\Lambda^k V$ can be derived in terms of those of V.

Theorem 5 (First Projection Formula). *Let* (V, ρ) *be a representation of* G. *Define the averaging map*

$$\phi := \frac{1}{|G|} \sum_{g \in G} \rho_g \in \mathrm{End}(V).$$

This is a homomorphism of representations, and a projection onto V^G. *In particular, therefore,*

$$\dim V^G = \operatorname{tr} \phi = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Proof. The first statement follows from the straightforward check that $\phi \circ \rho_h = \phi = \rho_h \circ \phi$ for all $h \in G$ (make sure you perform said check!). From this, it follows also that

$$\phi^2 = \phi \circ \left(\frac{1}{|G|} \sum_{g \in G} \rho_g) \right) = \frac{1}{|G|} \sum_{g \in G} \phi \circ \rho_g = \frac{1}{|G|} \sum_{g \in G} \phi = \phi.$$

It is also immediate to check that $\phi|_{V^G} = \mathrm{id}_{V^G}$; this along with $\phi^2 = \phi$ and $\mathrm{im}\, \phi \subseteq V^G$ is enough to conclude that ϕ is a projection onto V^G (make sure you recall the definition of a projection). It is easy to see (using a matrix, e.g.) that the trace of a projection is the dimension of its image, that the claim follows from the linear of the trace operator:

$$\dim V^G = \operatorname{tr} \phi = \operatorname{tr} \left(\frac{1}{|G|} \sum_{g \in G} \rho_g \right) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(\rho_g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

■

Corollary 5.1 (Burnside's Lemma/Cauchy-Frobenius Formula). *Let a finite group* G *act on a finite set* X, *and let* V *denote the corresponding permutation representation. Then for every* $g \in G$, *we have that* $\chi_V(g) = |X^g|$ *is the number of fixed points of* g. *Further,* $\dim V^G = |X/G|$ *is the number of orbits, so*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof. This is a permutation representation, so ρ_g is a permutation matrix (i.e. a matrix whose columns form some permutation of the columns of the identity matrix); in particular, the matrix of ρ_g consists only of 0's and 1's, and the number of 1's on the diagonal is exactly the number $|X^g|$ of elements fixed by g (make sure you understand this point!). Therefore, $\chi_V(g) = \operatorname{tr}(\rho_g) = |X^g|$. With the previous theorem at hand, it remains only to show that $\dim V^G$ is $|X/G|$, the number of orbits of the action. For any orbit $\mathcal{O} \in X/G$, define $\eta_{\mathcal{O}} := \sum_{x \in \mathcal{O}} e_x \in V$. We claim that $\{\eta_{\mathcal{O}}\}_{\mathcal{O} \in X/G}$ forms a basis of V^G. Firstly, since $\mathcal{O}$ is an orbit, the action of g on $\mathcal{O}$ induces a bijection $\mathcal{O} \xrightarrow{g \cdot} \mathcal{O}$, so that we have

$$\rho_g \eta_{\mathcal{O}} = \sum_{x \in \mathcal{O}} \rho_g e_x = \sum_{x \in \mathcal{O}} e_{g \cdot x} = \sum_{x \in \mathcal{O}} e_x = \eta_{\mathcal{O}};$$

this shows that each $\eta_{\mathcal{O}} \in V^G$. Since orbits of an action are disjoint, it is clear that the different $\{\eta_{\mathcal{O}}\}$ are linearly independent. Finally, we have to show that every element of V^G can be written as a linear combination of the $\eta_{\mathcal{O}}$: for that, it suffices to show that if $\eta \in V^G$, then the coefficient of e_x in η depends only on the orbit $\mathcal{O}$ of x. For that, suppose x and y belong to the same orbit, so that $\exists g \in G : gx = y$. Then the coefficient of e_x in η is the same as the coefficient of e_y in $\rho_g \eta = \eta$, as needed. ■

Remark 21. The direction of the above corollary can be pursued further to derive the unweighted and weighted Polyá Enumeration Theorems, which are powerful theorems in combinatorics.

Corollary 5.2. *Given representations* (V, ρ) *and* (W, π)*, we have that*

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}(V, W)^G = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g).$$

Definition 13. Give the space $\mathbb{C}^G$ the space of a Hermitian inner product space by defining the positive definite Hermitian inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}$ by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\alpha(g)}\beta(g).$$

In this new language, we get that $\dim \mathrm{Hom}_G(V, W) = \langle \chi_V, \chi_W \rangle$. From Corollary 0.1, the following result follows.

Corollary 5.3. *In terms of this inner product on* $\mathbb{C}^G$*, the characters of distinct irreducible representations of* G *are orthonormal.*

Let $\mathbb{C}^G_{cl} \subseteq \mathbb{C}^G$ denote the subspace of class functions, i.e. functions constant on conjugacy classes. Then $\dim \mathbb{C}^G_{cl} = |\mathscr{C}(G)|$ is the number of conjugacy classes of G.

Corollary 5.4. *If* G *is a finite group, then it has at most* $|\mathscr{C}(G)|$ *distinct irreducible representations.*

Proof. If (V, ρ) is any representation of G, then $\chi_V \in \mathbb{C}^G_{cl}$. Since orthonormal vectors are linearly independent (why?), the number of distinct irreducible representations of G is at most $\dim \mathbb{C}^G_{cl} = |\mathscr{C}(G)|$. ■

Corollary 5.5. *A representation of a group* G *is completely determined by its character.*

Proof. If (V, ρ) is any representation, then the number a_i of copies of an irreducible representation V_i in V is given by $\langle \chi_V, \chi_{V_i} \rangle$, and hence completely determined by χ_V. ■

Corollary 5.6. *If* (V, ρ) *is a representation of a group* G*, with decomposition* $\left(\bigoplus_{i=1}^n V_i^{\oplus a_i}, \bigoplus_{i=1}^n \rho_i^{\oplus a_i}\right)$ *into irreducible representations, then* $\chi_V = \sum_{i=1}^n a_i \chi_{V_i}$*, and therefore,*

$$\langle \chi_{V_i}, \chi_{V_i} \rangle = \sum_{i=1}^n a_i^2.$$

In particular, V *is irreducible iff* $\langle \chi_V, \chi_V \rangle = 1$*.*

Proof. This follows from Theorem 4 (2) and orthonormality. The second claim follows because 1 can only be written as a sum of squares as $1 = 1^2$. ■

Corollary 5.7. *If* (V, ρ) *is any representation of* G*, then TFAE:*

(1) (V, ρ) *is irreducible.*
(2) (V^*, ρ^*) *is irreducible.*
(3) For every one-dimensional representation (W, π) *of* G*, the tensor product* $(V \otimes W, \rho \otimes \pi)$ *is irreducible.*
(4) For some one-dimensional representation (W, π) *of* G*, the tensor product* $(V \otimes W, \rho \otimes \pi)$ *is irreducible.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is true because of the previous corollary and

$$\langle \chi_V, \chi_V \rangle = \frac{1}{|G|}\sum_{g\in G} |\chi_V(g)|^2 = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\chi_{V^*}(g) = \langle \chi_{V^*}, \chi_{V^*} \rangle.$$

If W is one-dimensional, then for every $g \in G$, the character $\chi_W(g) \in S^1$ is a root of unity, so that $|\chi_{V\otimes W}(g)|^2 = |\chi_V(g)\chi_W(g)|^2 = |\chi_V(g)|^2$, and so $\langle \chi_V, \chi_V \rangle = \langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle$, proving (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). ■

Theorem 6 (Decomposition of the Regular Representation). *If* (R, ρ) *denotes the left regular representation of* G*, then*

$$\chi_R(g) = \begin{cases} |G|, & g = e, \\ 0, & \textit{else.} \end{cases}$$

In particular, $\langle \chi_R, \chi_{V_i} \rangle = \chi_{V_i}(e) = \dim V_i$*, so that every irreducible representation* (V_i, ρ_i) *of* G *occurs in* R *with multiplicity* $\dim V_i$*.*

Proof. This follows from the discussion in Corollary 5.1, because the left regular representation of G is the permutation representation corresponding to the action of G on itself by left multiplcation: specifically, $\chi_R(g) = |G^g| = \#\{h \in G : gh = h\}$, from which the claim follows. Now

$$\langle \chi_R, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)} \chi_{V_i}(g) = \frac{1}{|G|} \overline{\chi_R(e)} \chi_{V_i}(e) = \dim V_i,$$

and we observed above that for any representation (V, ρ), the number $\langle \chi_V, \chi_{V_i} \rangle$ is the multiplicitiy of (V_i, ρ_i) appearing in (V, ρ). ■

Corollary 6.1. *If* G *is a finite group, then* $|G| = \sum_i (\dim V_i)^2$*, where the sum is over irreducible representations of* G*.*

Proof. If (R, ρ) denotes the regular representation, then from Corollary 5.6 and the previous theorem, we ge that

$$|G| = \frac{1}{|G|} \overline{\chi_R(e)} \chi_R(e) = \langle \chi_R, \chi_R \rangle = \sum_i (\dim V_i)^2.$$

■

Corollary 6.2. *If* G *is an abelian group, then it has exactly* $|G|$ *distinct irreducible representations.*

Proof. We know that all irreducible representations of an abelian group are 1-dimensional, so that $|G| = \sum_i (\dim V_i)^2$ tells us that there are $|G|$ of them. ■

Given a group G, a representation (V, ρ) of G, and a function $\alpha \in \mathbb{C}^G$, we get the map

$$\phi_{\alpha,V} := \frac{1}{|G|} \sum_{g \in G} \alpha(g) \rho_g \in \mathrm{End}(V).$$

We have seen that

(1) if $\alpha = |G| 1_g$ for some $g \in G$, then $\phi_{\alpha,V}$ is G-linear iff $[g, h] = ghg^{-1}h^{-1} \in \ker \rho$ for all $h \in G$.
(2) if $\alpha \equiv 1$, then $\phi_{\alpha,V}$ is always G-linear.

We now generalize these, and ask the question about G-linearity of $\phi_{\alpha,V}$ in general.

Theorem 7. *The map* $\phi_{\alpha,V}$ *is* G*-linear for all* (V, ρ) *iff* $\alpha \in \mathbb{C}^G_{cl}$*.*

Proof. If $\alpha \in \mathbb{C}^G_{cl}$, then for all $h \in G$ and $v \in V$, we have

$$\begin{aligned} (\phi_{\alpha,V} \circ \rho_h)(v) &= \frac{1}{|G|} \sum_{g \in G} \alpha(g) \rho_g \rho_h v \\ &= \frac{1}{|G|} \sum_{g \in G} \alpha(hgh^{-1}) \rho_{hgh^{-1}} \rho_h v \\ &= \rho_h \left(\frac{1}{|G|} \sum_{g \in G} \alpha(g) \rho_g v \right) = (\rho_h \circ \phi_{\alpha,V})(v). \end{aligned}$$

Conversely, suppose that $\phi_{\alpha,V}$ is G-linear for all (V, ρ). Then in particular, it is for the regular representation (R, ρ). This means that for the basis element $e_e \in R$ and a fixed $h \in G$, we have

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}\alpha(g)e_{hg} &= \rho\left(\frac{1}{|G|}\sum_{g\in G}\alpha(g)\rho_g e_e\right)\\ &= (\rho_h \circ \phi_{\alpha,R})(e_e)\\ &= (\phi_{\alpha,R} \circ \rho_h)(e_e)\\ &= \frac{1}{|G|}\sum_{g\in G}\alpha(hgh^{-1})\rho_{hgh^{-1}}\rho_h e_e = \frac{1}{|G|}\sum_{g\in G}\alpha(hgh^{-1})e_{hg}.\end{aligned}$$

Since in R the elements $\{e_g : g \in G\}$ form a basis by definition, so that this means that $\alpha(g) = \alpha(hgh^{-1})$ for all $g \in G$. Since this is true of all $h \in G$, we have $\alpha \in \mathbb{C}^G_{cl}$. ■

Corollary 7.1. *The characters $\{\chi_{V_i}\}$ of distinct irreducible representations of G form a basis for $\mathbb{C}^G_{cl}$. In particular, the group G has exactly $|\mathscr{C}(G)|$ distinct irreducible representations.*

Proof. Since the characters $\{\chi_{V_i}\}$ of irreducible representations are orthonormal, by Gram-Schmidt orthonormalization, we can extend this to an orthonormal basis of $\mathbb{C}^G_{cl}$ (which, remember, has finite dimension $|\mathscr{C}(G)|$). To prove the result, it suffices to show that if $\alpha \in \mathbb{C}^G_{cl}$ is orthogonal to all the $\{\chi_{V_i}\}$, then it is zero (why?). For that, let $\alpha \in \mathbb{C}^G_{cl}$ be orthogonal to all the $\{\chi_{V_i}\}$. Let (V_i, ρ_i) be an irreducible representation, and consider the G-linear map $\phi_{\alpha,V_i} : V_i \to V_i$ as above. By Schur's Lemma, this is $\lambda\, \mathrm{id}_{V_i}$ for some $\lambda \in \mathbb{C}$. Then

$$\lambda \dim V_i = \mathrm{tr}(\phi_{\alpha,V_i}) = \frac{1}{|G|}\sum_{g\in G}\alpha(g)\chi_V(g) = \langle \alpha, \chi_{V_i^*}\rangle = 0,$$

since V_i^* is irreducible too. This tells us that $\lambda = 0$, so that in fact $\phi_{\alpha,V_i} = 0 : V_i \to V_i$ is the zero map. Since this is true of every irreducible representation V_i, complete reducibility tells us that this is true of every finite dimensional representation of G (how?). In particular, it is true for the regular representation (R, ρ) of G. But then

$$0 = \phi_{\alpha,R}(e_e) = \frac{1}{|G|}\sum_{g\in G}\alpha(g)\rho_g e_e = \frac{1}{|G|}\sum_{g\in G}\alpha(g)e_g.$$

But the elements $e_g \in R$ are linearly independent, so this means that $\alpha(g) = 0$ for all $g \in G$, i.e. $\alpha = 0$. ■

Corollary 7.2. *The complexification $\chi_{\mathbb{C}} : R_{\mathbb{C}}(G) \otimes \mathbb{C} \to \mathbb{C}^G_{cl}$ is a ring isomorphism.*

Proof. This is an injective $\mathbb{C}$-linear map between $\mathbb{C}$-vector spaces of the same dimension. (Explore why each statement made in this claim is true.) ■

Theorem 8 (General Projection Formula). *Consider the special case of the above where $\alpha = \chi_{V_i^*}$ and $V = V_j$ for V_i, V_j irreducible representations. By Schur's Lemma, $\phi_{\chi_{V_i^*},V_j} : V_j \to V_j$ is $\lambda\, \mathrm{id}_{V_j}$ for some $\lambda \in \mathbb{C}$. This λ can be found by*

$$\lambda = \frac{1}{(\dim V_j)\cdot |G|}\sum_{g\in G}\overline{\chi_{V_i}}\chi_{V_j}(g) = \frac{1}{(\dim V_j)\cdot |G|}\langle \chi_{V_i}, \chi_{V_j}\rangle.$$

This is nonzero iff $i = j$; and if $i = j$, then this is a nonzero multiple of the identity. Therefore, for an arbitrary representation the map

$$(\dim V_i)\cdot |G|\phi_{\chi_{V_i^*},V} : V \to V$$

is the projection onto the $V_i^{\oplus a_i}$ component.

Proof. This follows immediately from the previous discussion. See Fulton and Harris p. 23. ■

6 Induced Representations

This section is going to be short, since it was covered most recently in class. If $H \subseteq G$ is a subgroup, then get a functor Res_H^G from the representations of G to the representations of H by restriction. In the other direction, we have the induction functor Ind_H^G.

Definition 14. Suppose we have a representation (V, ρ) of G, and we have a subrepresentation $(W, \rho|_W) \subseteq \mathrm{Res}_H^G(V, \rho)$. Then (V, ρ) is said to be *induced* by $(W, \rho|_W)$ if as a vector space we have $V = \bigoplus_{\sigma \in G/H} \sigma W$. In this case, we write $(V, \rho) = \mathrm{Ind}_H^G(W, \rho|_W)$.

Theorem 9. *Given a representation* (W, π) *of* H*, the induced representation* $(V, \rho) := \mathrm{Ind}_H^G(W, \pi)$ *exists and is unique upto isomorphism of* G*-representations. Further,* $\dim V = |G/H| \dim W$*.*

Proof Sketch. For existence, take a copy $(W, \pi)^\sigma$ for each $\sigma \in G/H$. For each σ, pick a representative $g_\sigma \in G$ of it, and let $g_\sigma w \in W^\sigma$ denote the element corresponding to $w \in W$. The action by G is by permuting the left cosets. The claim about dimension follows immediately. ■

Theorem 10 (Frobenius Reciprocity). *f* (U, τ) *is any other representation of* G*, then an* H*-linear map* $(W, \pi) \to \mathrm{Res}_H^G(U, \tau)$ *extends uniquely to* G*-linear map* $\mathrm{Ind}_H^G(W, \pi) \to (U, \tau)$*, so that*

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G W, U) \cong \mathrm{Hom}_H(W, \mathrm{Res}_H^G U).$$

In particular,

$$\langle \chi_{\mathrm{Ind}_H^G W}, \chi_U \rangle_G = \langle \chi_W, \chi_{\mathrm{Res}_H^G U} \rangle_H.$$

If W *and* U *are irreducible, then the number of times* U *appears in* $\mathrm{Ind}_H^G W$ *is the same as the number of times* W *appears in* $\mathrm{Res}_H^G U$*.*

Proof Sketch. Every map $\mathrm{Ind}_H^G W \to U$ restricts to a map $eW = W \to \mathrm{Res}_H^G U$. Conversely, every H-linear map $\phi : (W, \pi) \to \mathrm{Res}_H^G(U, \tau$ extends uniquely to a G-linear map $\mathrm{Ind}_H^G(W, \pi) \to (U, \tau)$ via the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi} & U \\ \big\downarrow{\scriptstyle g_\sigma} & & \big\downarrow{\scriptstyle \tau_{g_\sigma}} \\ W^\sigma & \xrightarrow{\exists!\, \widetilde{\phi}} & U. \end{array}$$

Comparing dimensions gives the relation between characters. If W and U are irreducible, then the LHS in the character formula is the number of times U appears in $\mathrm{Ind}_H^G W$, and the RHS is the number of times W appears in $\mathrm{Res}_H^G U$. ■

Theorem 11. *If* (W, π) *is any representation of* $H \leq G$*, then for any* $\gamma \in G$*, we have that*

$$\chi_{\mathrm{Ind}_H^G W}(\gamma) = \sum_{\substack{\sigma \in G/H \\ \gamma\sigma = \sigma}} \chi_W(g_\sigma^{-1} \gamma g_\sigma),$$

where $g_\sigma \in G$ *is some representative of* σ*.*

Proof Sketch. If (V, ρ) denotes $\mathrm{Ind}_H^G(w, \pi)$, then for any $\gamma \in G$, ρ_γ maps $\sigma W \to \gamma\sigma W$. If the cosets σ and $\gamma\sigma$ are different, then there is no term corresponding to this on the diagonal; if they are the same, then $\rho_\gamma : \sigma W \to \sigma W$ is given the following composition

$$\sigma W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{g_\sigma^{-1}\gamma g_\sigma} W \xrightarrow{g_\sigma^{-1}} \sigma W,$$

which has the same trace as the central map. Therefore, the contribution of this block is exactly $\chi_W(g_\sigma^{-1} \gamma g_\sigma)$. ■

Remark 22. Recall, as Professor Auroux mentioned in class, that it is not true that $\chi_W(g_\sigma^{-1}\gamma g_\sigma) = \chi_W(\gamma)$, because the latter doesn't even make sense in general–χ_W is only defined on H. The conjugation is needed to get the element $g_\sigma^{-1}\gamma g_\sigma \in H$ to which we *can* apply χ_W.

7 Example: $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$

Let's now turn to a concrete example: let's figure out the character table for the group $G := \mathbb{Z}/7 \rtimes \mathbb{Z}/3$, the unique nonabelian group of order 21. Recall from class that this has presentation

$$G = \langle a, b \mid a^7 = b^3 = e, bab^{-1} = a^2 \rangle.$$

In this group, the subgroup $\langle a \rangle \leq G$ is a normal subgroup of order 7, with quotient $G/\langle a \rangle \cong \langle b \rangle \cong \mathbb{Z}/3$. The last relation $bab^{-1} = a^2$ tells us that $b^k a^\ell = a^{2^k \ell} b^k$ for any $k \in \mathbb{Z}/3, \ell \in \mathbb{Z}/7$, so that every element of G can be written uniquely in the form $a^i b^j$ for $0 \leq i \leq 6$ and $0 \leq j \leq 2$. Let's first figure out the conjugacy structure of G.

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(1) Let's figure out the center $Z(G)$. Suppose $g = a^i b^j \in Z(G)$ for some $0 \leq i \leq 6$ and $0 \leq j \leq 2$. Then
$$a^i b^j = g = aga^{-1} = a^{i+1} b^j a^{-1} = a^{i+1-2^j} b_j,$$
so that $i \equiv i + 1 - 2^j \pmod 7$, so $2^j \equiv 1 \pmod 7$. Since $0 \leq j \leq 2$, this tells us that $j = 0$, and $g = a^i$ for some $0 \leq i \leq 6$. Then
$$a^i = g = bgb^{-1} = ba^i b^{-1} = a^{2i},$$
so that $i \equiv 2i \pmod 7$ and hence $i = 0$. This means that $Z(G) = \{e\}$, i.e. the center of G consists of the identity alone.

(2) This tells us that the class equation of G looks like
$$21 = 1 + \sum_{i=1}^{r} |K_i|$$
for some $r \geq 1$ such that $2 \leq |K_i| \mid 21$ for all i. This means that all the $|K_i| \in \{3, 7\}$. The only way to write 20 as a sum of 3's and 7's is $20 = 3 + 3 + 7 + 7$, so that we must have $r = 4$. The class equation looks like
$$21 = 1 + 3 + 3 + 7 + 7.$$
From this, we know that there are exactly 5 irreducible representations of G.

(3) Let's try to figure out the elements in the conjugacy classes. The relations $bab^{-1} = a^2$ tells us that the conjugacy class of a is $\{a, a^2, a^4\}$. By symmetry, the other conjugacy class of size 3 must be $\{a^{-1} = a^6, a^5, a^3\}$. Finally, the relation $b^k a^\ell = a^{2^k \ell} b^k$ tells us that conjugating by either a or b doesn't change the exponent of b in the element, so that it is not hard to see that the remaining 2 conjugacy classes must be $\{a^i b\}_{i=0}^6$ and $\{a^i b^2\}_{i=0}^6$.

Now let's try to figure out the 5 irreducible representations of G. Of course, we know the first one–it's the 1-dimensional trivial representation. Let's start writing the character table:

$\mathbb{Z}/7 \rtimes \mathbb{Z}/3$	1	3	3	7	7
$\langle a, b \mid a^7, b^3, bab^{-1}a^{-2} \rangle$	e	a	a^{-1}	b	b^{-1}
U	1	1	1	1	1

Now observe that we have a normal subgroup $\mathbb{Z}/7 \cong \langle a \rangle \trianglelefteq G$ with quotient $G/\langle a \rangle \cong \langle b \rangle \cong \mathbb{Z}/3$. Now $\mathbb{Z}/3$ has exactly three 1-dimensional irreducible representations: those in which b acts by $1, \omega$, and ω^2 respectively, where $\omega = \exp(2\pi i/3)$ is a complex cube root of unity. Pulling these back to G gives us two more 1-dimensional, and hence irreducible, representations of G with characters $(1, 1, 1, \omega, \omega^2)$ and $(1, 1, 1, \omega^2, \omega)$ respectively. Therefore, in total we now know 3 of 5 irreducible representations of G:

$\mathbb{Z}/7 \rtimes \mathbb{Z}/3$	1	3	3	7	7
$\langle a, b \mid a^7, b^3, bab^{-1}a^{-2} \rangle$	e	a	a^{-1}	b	b^{-1}
U	1	1	1	1	1
U'	1	1	1	ω	ω^2
U''	1	1	1	ω^2	ω

We know we're missing two more representations, say V and V' of dimensions d and d' respectively. Then

$$21 = 1^2 + 1^2 + 1^2 + d^2 + d'^2$$

tells us that $d = d' = 3$. Suppose that $\chi_V = (3, p, q, r, s)$. Then

$$\langle \chi_U, \chi_V \rangle = \frac{1}{21}(3 + 3p + 3q + 7r + 7s) = 0.$$

Similarly, we also get that $3 + 3p + 3q + 7\omega^2 r + 7\omega s = 3 + 3p + 3q + 7\omega r + 7\omega^2 s = 0$. Adding these three equations together, we get that $1+p+q = 0$ and $r+s = r+\omega s = r+\omega^2 s = 0$, so that $q = -1-p$ and $r = s = 0$. Since the same is true of V', the table now looks like the following for some $\alpha, \beta \in \mathbb{C}$.

$\mathbb{Z}/7 \rtimes \mathbb{Z}/3$	1	3	3	7	7
$\langle a, b \mid a^7, b^3, bab^{-1}a^{-2} \rangle$	e	a	a^{-1}	b	b^{-1}
U	1	1	1	1	1
U'	1	1	1	ω	ω^2
U''	1	1	1	ω^2	ω
V	3	α	$-1-\alpha$	0	0
V'	3	β	$-1-\beta$	0	0

Now since V is an irreducible representation, so is V^*. We are reduced to two possibilities:

(1) If $V \cong V^*$, then $V' \cong V'^*$ (why?), and $\alpha \in \mathbb{R}$. Using the orthonormality relations, we get the quadratic equation $\alpha^2 + \alpha - 3/2 = 0$, so that $\alpha \in \left\{\frac{-1\pm\sqrt{7}}{2}\right\}$. WLOG, suppose $\alpha = \frac{-1+\sqrt{7}}{2}$; then we must have $\beta = \frac{-1-\sqrt{7}}{2}$, and all symmetry relations are satisfied.

(2) If $V \not\cong V^*$, then we must have $V^* \cong V'$ and $V'^* \cong V$. This tells us that $\beta = \overline{\alpha}$, and using the orthonormality relations we get to $\alpha^2 + \alpha + 2 = 0$, so that $\alpha \in \left\{\frac{-1\pm\sqrt{7}i}{2}\right\}$. WLOG, suppose $\alpha = \frac{-1+\sqrt{7}i}{2}$; then we must have $\beta = \frac{-1-\sqrt{7}i}{2}$, and again all symmetry relations are satisfied.

How do we decide between these two possibilities?

The key to answering this question is Frobenius Reciprocity. Recall that we have a normal subgroup $H := \langle a \rangle$ of G of order 7. This has exactly seven 1-dimensional irreducible representations: U_i for $0 \le i \le 6$, with a acting on U_i as ζ_7^i, where $\zeta_7 = \exp(2\pi i/7)$ is a primitive 7^{th} root of unity.

(1) Now $\mathrm{Res}_H^G V$ has character $(3, \alpha, \alpha, -1-\alpha, \alpha, -1-\alpha, -1-\alpha)$ in the usual order. In particular, since $\alpha \neq 3$, this must contain some U_i for $i \neq 0$. Since U_i appears in $\mathrm{Res}_H^G V$, Frobenius reciprocity tells us that V appears in $\mathrm{Ind}_H^G U_i$. But now

$$\dim V = 3 = |G/H| \dim U_i = \dim \mathrm{Ind}_H^G U_i,$$

so that V must equal one of the $\mathrm{Ind}_H^G U_i$ for some $i \neq 0$. Therefore, it suffices to analyze the $\mathrm{Ind}_H^G U_i$.

(2) From class, we know that

$$\chi_{\mathrm{Ind}_H^G U_i}(a) = \sum_{\substack{\sigma \in G/H \\ a\sigma = \sigma}} \chi_{U_i}(g_\sigma^{-1} a g_\sigma) = \chi_{U_i}(a) + \chi_{U_i}(b^{-1}ab) + \chi_{U_i}(bab^{-1}) = \zeta_7^i + \zeta_7^{4i} + \zeta_7^{2i}.$$

As i varies over $0 \le i \le 6$, the expression $\zeta_7^i + \zeta_7^{2i} + \zeta_7^{4i}$ takes only two distinct values, namely $\zeta_7 + \zeta_7^2 + \zeta_7^4$ and $\zeta_7^6 + \zeta_7^5 + \zeta_7^3$. These two, then, must be α and β in some order. Therefore, we conclude WLOG that $\alpha = \zeta_7 + \zeta_7^2 + \zeta_7^4$ and $\beta = \zeta_7^6 + \zeta_7^5 + \zeta_7^3$.

(3) Wait–how do we reconcile this with our previous answer? Well, α and β are complex conjugates, so that we must be in case (2) above. It is easy to see (by drawing a picture, e.g.) that $\zeta_7+\zeta_7^2+\zeta_7^4$ has positive imaginary part, so that we must have

$$\alpha = \zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{-1+\sqrt{7}i}{2}, \text{ and } \beta = \zeta_7^6 + \zeta_7^5 + \zeta_7^3 = \frac{-1-\sqrt{7}i}{2}.$$

Remark 23. These identities involving ζ_7 can also be proved by adding together and multiplying α and β, and then simplifying using the fact that $\sum_{i=0}^{6} \zeta_7^i = 0$. These are in fact quadratic Gauss sums–3 is a primitive root modulo 7–and the algebra closely mimics the geometry of the situation with $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ being the group of affine transformations of $\mathbb{A}^1(\mathbb{F}_7)$ that dilate by a cube root of unity in $\mathbb{F}_7$.

This allows us to complete the character table:

$\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ $\langle a, b \vert a^7, b^3, bab^{-1}a^{-2} \rangle$	1 e	3 a	3 a^{-1}	7 b	7 b^{-1}
U	1	1	1	1	1
U'	1	1	1	ω	ω^2
U''	1	1	1	ω^2	ω
V	3	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	$\zeta_7^6 + \zeta_7^5 + \zeta_7^3$	0	0
V'	3	$\zeta_7^6 + \zeta_7^5 + \zeta_7^3$	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	0	0

We have therefore characterized the five irreducible representations of $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$–these are the three irreducible representations of $\mathbb{Z}/3$ pulled back via the quotient $G \to G/H$, and the two irreducible representations induced by the nontrivial representations of H.