Some Genera Computations

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Abstract

This paper is written in fulfillment of the requirements of Math 293X: Topological Modular Forms taught by Dr. Stephen McKean at Harvard in the Fall 2023 semester. In this paper, we explain the general theory of multiplicative sequences and genera as laid out by Hirzebruch in his seminal textbook [1], following which we present explicit computations of total K-classes and K-genera for some multiplicative sequences K on a few closed complex manifolds.

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1 Multiplicative Sequences

Definition 1.0.1. Let R be a (commutative unitary) ring, and let $R[c]$ be the graded R-algebra defined by

$$R[c] := R[c_1, c_2, \dots], \quad \text{where for } n \geq 1, \text{we have } \deg c_n = 2n.[1]$$

It is often convenient to let $c_0 := 1$. Note that for each $n \geq 0$, the degree $2n$ component of $R[c]$ is given by

$$R[c]_{2n} = R[c_1, \dots, c_n]_{2n}.$$

Let $\mathrm{U}_1(R)$ be the group defined by

$$\mathrm{U}_1(R) := 1 + zR[\![z]\!] := \left\{ \sum_{n=0}^{\infty} q_n z^n : q_n \in R, q_0 = 1 \right\},$$

with group operation the multiplication of power series.

- A sequence $K = (K_n)_{n\geq 0}$ of elements of $R[c]$ is said to be multiplicative if $K_0 = 1$ and $K_n \in R[c]_{2n}$, and the map

$$K : \sum_{n=0}^{\infty} q_n z^n \mapsto \sum_{n=0}^{\infty} K_n(q_1, \dots, q_n) z^n$$

  is an endomorphism of the group $\mathrm{U}_1(R)$.
- Given a multiplicative sequence K, we define its characteristic series Q_K by

$$Q_K(z) := K(1 + z) = \sum_{n=0}^{\infty} K_n(1, 0, \dots, 0) z^n \in R_1.$$

Lemma 1.0.2. The map $K \mapsto Q_K(z)$ is a bijection from the set of all multiplicative sequences to $\mathrm{U}_1(R)$.

Proof. We define an inverse map. For each $N \geq 1$, let

$$R[c]^{(N)} := R[c][\gamma_1, \dots, \gamma_N]/(c_j - \sigma_j(\gamma_1, \dots, \gamma_N))_{j=1}^{N},$$

where $\sigma_j(\gamma_1, \dots, \gamma_N)$ is the j^{th} elementary symmetric polynomial in $\gamma_1, \dots, \gamma_N$. Let $Q(z) \in \mathrm{U}_1(R)$ be given. Consider the product

$$\prod_{i=1}^{N} Q(\gamma_i z) \in \mathrm{U}_1\left(R[c]^{(N)}\right).$$

Since the coefficients of the powers of z in the expansion of this product are symmetric in the γ_j, by the Fundamental Theorem of Symmetric Polynomials, they can be written as polynomials in the c_j. Therefore, for $n \geq 0$, there are $K_n^{(N)} \in R[c]_{2n}$ such that

$$\prod_{i=1}^{N} Q(\gamma_i z) = \sum_{n=0}^{\infty} K_n^{(N)}(c_1, \dots, c_n) z^n.$$

[1] This grading is not the same as the one chosen by Hirzebruch in [1, §1]. The reason for our choice is that for any complex vector bundle $E \to X$, the evaluation at the Chern classes of E gives us a ring homomorphism $\mathrm{eval}_E : R[c] \to \mathrm{H}^*(X; R)$, which with our convention becomes a morphism of graded R-algebras. See §2.

For $N \geq n$, the polynomial $K_n^{(N)}$ is independent of N, and we define K_n to be this common value. It is then straightforward to check that the resulting sequence $K = (K_n)_{n\geq 0}$ is multiplicative, and that these operations give us inverse bijections. For more details, see [1, §1].

■

For low degrees, we can write this correspondence out explicitly: if

$$Q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots,$$

then we have

$$\begin{aligned} K_1 &= q_1 c_1, \\ K_2 &= q_2 c_1^2 + (-2q_2 + q_1^2) c_2, \\ K_3 &= q_3 c_1^3 + (-3q_3 + q_1 q_2) c_1 c_2 + (3q_3 - 3q_1 q_2 + q_1^3) c_3, \end{aligned}$$

and so on.

Example 1.0.3. The sequence $K = c$ defined by $K_n = c_n$ is called the identity sequence, since the corresponding endomorphism K is the identity map on $\mathrm{U}_1(R)$. Its characteristic series is $Q(z) = 1 + z$.

Definition 1.0.4. An element $Q(z) \in \mathrm{U}_1(R)$ is said to be even if $Q(z)$ only consists of even powers of z, i.e. for each $n \geq 0$, the coefficient $[z^{2n+1}]Q(z)$ of z^{2n+1} in $Q(z)$ is zero, or equivalently when there is a (necessarily unique) $\widetilde{Q}(z) \in \mathrm{U}_1(R)$ such that $Q(z) = \widetilde{Q}(z^2)$.

If $Q(z)$ is even, then in the corresponding multiplicative sequence (K_n), we also have for each $n \geq 0$ that $K_{2n+1} = 0$. In this case, the sequence of polynomials $(\widetilde{K}_n)_{n\geq 0}$ defined by $\widetilde{K}_n = K_{2n}$ is called the corresponding reduced sequence.[2]

Example 1.0.5. The reduced sequence corresponding to $Q(z) = 1 + z^2$ is called the Pontryagin sequence and is denoted by (p_n). Explicitly, we have for $n \geq 0$ that

$$p_n = c_n^2 - 2\sum_{k=1}^{n} (-1)^k c_{n-k} c_{n+k}.$$

Again by the Fundamental Theorem of Symmetric Polynomials, this time applied to the γ_j^2, we conclude that if $Q(z)$ is any even series and $(\widetilde{K}_n)$ the corresponding reduced sequence, then for each $n \geq 0$, we have

$$\widetilde{K}_n \in R[p_1, \ldots, p_n]_{4n},$$

i.e. $\widetilde{K}_n$ can be written as a polynomial of total degree $4n$ in the (weighted variables) $p_1, \ldots, p_n$.

Suppose for the next three examples that R is a $\mathbb{Q}$-algebra.

[2] Note that this is not multiplicative in the above sense, and indeed $\deg \widetilde{K}_n = 4n$.

Example 1.0.6. Consider

$$Q_{\mathrm{Td}}(z) := \frac{z}{1-\mathrm{e}^{-z}} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1 + \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{120}z^4 + \frac{1}{30240}z^6 + \cdots,$$

the exponential generating function of the Bernoulli numbers B_n.[3] The corresponding multiplicative sequence is called the sequence of Todd polynomials Td_n, and the first few of these are given by

$$\begin{aligned}
\mathrm{Td}_0 &= 1 \\
\mathrm{Td}_1 &= \frac{1}{2}c_1, \\
\mathrm{Td}_2 &= \frac{1}{12}(c_1^2 + c_2), \\
\mathrm{Td}_3 &= \frac{1}{24}c_1c_2, \\
\mathrm{Td}_4 &= \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4).
\end{aligned}$$

Here's a fun observation:

Lemma 1.0.7. The series $Q_{\mathrm{Td}}(z)$ is uniquely characterized by the property that for any $n \geq 0$, the coefficient of z^n in $Q_{\mathrm{Td}}(z)^{n+1}$ is 1. In particular, the substitution $c_i := \binom{n+1}{i}$ in Td_n yields

$$\mathrm{Td}_n(c_1, \dots, c_n) = 1.$$

Proof. The existence and uniqueness of such a series is inductively clear, so it suffices to show that $Q_{\mathrm{Td}}(z)$ satisfies this property. Now this coefficient is given by

$$[z^n]Q_{\mathrm{Td}}(z)^{n+1} = [z^{-1}]\frac{1}{(1-\mathrm{e}^{-z})^{n+1}} = \operatorname*{Res}_{z=0} \frac{1}{(1-\mathrm{e}^{-z})^{n+1}}\mathrm{d}z.$$

To compute this residue, write $w := 1 - \mathrm{e}^{-z}$, which is a holomorphic change of coordinates around $z = 0$, to get

$$\operatorname*{Res}_{z=0} \frac{1}{(1-\mathrm{e}^{-z})^{n+1}}\mathrm{d}z = \operatorname*{Res}_{w=0}\left(\frac{1}{w^{n+1}} \cdot \frac{\mathrm{d}w}{1-w}\right) = 1.$$

For the second result, note that if $c_i = \binom{n+1}{i}$, then from

$$1 + c_1 z + \cdots + c_n z^n \equiv (1+z)^{n+1} \pmod{z^{n+1}},$$

it follows that

$$\begin{aligned}
1 + \mathrm{Td}_1(c_1)z + \cdots + \mathrm{Td}_n(c_1, \dots, c_n)z^n &\equiv \mathrm{Td}(1 + c_1 z + \cdots + c_n z^n) \\
&\equiv \mathrm{Td}(1+z)^{n+1} \\
&\equiv Q_{\mathrm{Td}}(z)^{n+1} \pmod{z^{n+1}},
\end{aligned}$$

so equating the coefficients of z^n on both sides yields $\mathrm{Td}_n(c_1, \dots, c_n) = 1$. ∎

[3] Here we are choosing the "positive" convention $B_1 = 1/2$.

Example 1.0.8. Consider

$$Q_L(z) := \frac{z}{\tanh z} = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n} = 1 + \frac{1}{3}z^2 - \frac{1}{45}z^4 + \frac{2}{945}z^6 + \cdots .$$

The corresponding reduced sequence is called the sequence of L-polynomials and denoted (L_n). The first few of these are given by

$$\begin{aligned}
L_0 &= 1 \\
L_1 &= \frac{1}{3}p_1, \\
L_2 &= \frac{1}{45}(-p_1^2 + 7p_2), \\
L_3 &= \frac{1}{945}(2p_1^3 - 13p_1p_2 + 62p_3), \\
L_4 &= \frac{1}{14175}(-3p_1^4 + 22p_1^2p_2 - 19p_2^2 - 71p_1p_3 + 831p_4).
\end{aligned}$$

The series

$$\widetilde{Q}_L(z) = \frac{\sqrt{z}}{\tanh \sqrt{z}} := \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^n$$

is uniquely characterized by the property that for any $n \geq 0$, the coefficient of z^n in $\widetilde{Q}_L(z)^{2n+1}$ is 1. Indeed, the proof is almost identical to that of Lemma 1.0.7; the interested reader can see [1, §1].

Example 1.0.9. Consider

$$Q_{\hat{A}}(z) := \frac{z/2}{\sinh(z/2)} = 1 - \frac{1}{24}z^2 + \frac{7}{5760}z^4 - \frac{31}{967680}z^6 + \cdots .$$

The corresponding reduced sequence is called the sequence of $\hat{A}$-polynomials and denoted $(\hat{A}_n)$. The first few of these are given by

$$\begin{aligned}
\hat{A}_0 &= 1 \\
\hat{A}_1 &= -\frac{1}{24}p_1, \\
\hat{A}_2 &= \frac{1}{5760}(7p_1^2 - 4p_2), \\
\hat{A}_3 &= \frac{1}{967680}(-31p_1^3 + 44p_1p_2 - 16p_3), \\
\hat{A}_4 &= \frac{1}{464486400}(381p_1^4 - 904p_1^2p_2 + 208p_2^2 + 512p_1p_3 - 192p_4).
\end{aligned}$$

2 Genera Computations

Let X be a (paracompact, Hausdorff) topological space. For any ring R, let $\mathrm{H}^*(X;R)$ denote the singular cohomology of X with coefficients in R. Associated to each complex vector bundle $E \to X$ and integer $n \geq 0$, we have the n^{th} Chern class of E with R-coefficients

$$c_n(E,R) \in \mathrm{H}^{2n}(E,R).$$

Evaluation at these Chern classes yields a graded R-algebra homomoprhism

$$\mathrm{eval}_E : R[c] \to \mathrm{H}^*(X;R).$$

By functoriality of the U_1 construction, we also get an evaluation map

$$\mathrm{eval}_E : \mathrm{U}_1(R[c]) \to \mathrm{U}_1(\mathrm{H}^*(X;R)). \tag{1}$$

Definition 2.0.1. Let $K = (K_n)$ be a multiplicative sequence in R. The total K-series of a vector bundle E, denoted $K(E,z) \in \mathrm{U}_1(\mathrm{H}^*(X;R))$ is the image of the series $\sum_{n=0}^{\infty} K_n z^n \in \mathrm{U}_1(R[c])$ under the evaluation map (1).

Example 2.0.2. If $K = c$ is the identity sequence, then the total K-series of E is nothing but the Chern polynomial $c(E,z) = \sum_{n=0}^{\infty} c_n(E,R) z^n \in \mathrm{U}_1(\mathrm{H}^*(X;R))$.[4]

Note that the Whitney formula for the Chern classes along with the multiplicativity of K implies that for any short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles on X, we have the product formula

$$K(E,z) = K(E',z)K(E'',z) \in \mathrm{U}_1(\mathrm{H}^*(X;R)). \tag{2}$$

This, along with the splitting principle, gives us a recipe to compute $K(E,z)$: indeed, suppose that $\operatorname{rank} E = r \geq 0$, and we formally factor the Chern polynomial as $c(E,z) = \prod_{i=1}^{r}(1+\gamma_i z)$, where the γ_i are the Chern roots of E. Then (2) tells us that if the multiplicative sequence $K = (K_n)$ has characteristic series $Q(z)$, then the total K-series of E can be written as

$$K(E,z) = \prod_{i=1}^{r} Q(\gamma_i z).$$

Here, as before, this means that the coefficients of the powers of z in this product, being symmetric in the γ_i, can be written as elementary symmetric polynomials in the γ_i, which are the Chern classes of E.

If X has cohomology ring $\mathrm{H}^*(X;\mathbb{Z})$ of finite rank over $\mathbb{Z}$ as an abelian group (e.g. if X is a closed manifold or a finite CW complex), then the evaluation of $K(E,z) \in \mathrm{U}_1(\mathrm{H}^*(X;R))$ at specific $z = z_0 \in R$ makes sense and gives an element $K(E,z_0) \in \mathrm{H}^*(X;R)$.[5]

[4] The name "Chern polynomial" is apt because $c_n(E,R) \neq 0$ only for finitely many n, namely those satisfying $0 \leq n \leq \operatorname{rank} E$.

[5] If $\mathrm{H}^*(X;\mathbb{Z})$ does not have finite rank, then this element lies rather in the direct product $\mathrm{H}^{\Pi}(X;R) = \prod_{k=0}^{\infty} \mathrm{H}^k(X;R)$ instead.

Definition 2.0.3. The total K-class of E, denoted $K(E)$, is simply the total series $K(E,z)$ of E evaluated at $z=1$.

Example 2.0.4. From the examples in §1, if $E \to X$ has Chern roots γ_i, then we have:

(a) the total Chern series $c(E,z) = \prod_i (1+\gamma_i z)$,
(b) the total Pontryagin series $p(E,z) = \prod_i (1+\gamma_i^2 z^2)$,
(c) the total Todd series $\mathrm{Td}(E,z) = \prod_i \gamma_i z/(1-\mathrm{e}^{-\gamma_i z})$,
(d) the total L-series $L(E,z) = \prod_i \gamma_i z/\tanh(\gamma_i z)$, and
(e) the total $\hat{A}$-series $\hat{A}(E,z) = \prod_i (\gamma_i z/2)/\sinh(\gamma_i z/2)$.

Evaluating these at $z=1$ yields, the total Chern, Pontryagin, Todd, L and $\hat{A}$ classes respectively.

Now suppose that X is a closed oriented manifold of dimension $N=2n$ and that $R=\mathbb{Z}$. Then $\mathrm{H}^N(X;\mathbb{Z}) = \mathbb{Z}\eta_X$ is a free $\mathbb{Z}$-module of rank 1 generated by the class η_X which is the algebraic dual to the fundamental class $[X] \in \mathrm{H}_N(X;\mathbb{Z})$, i.e. evaluation at the fundamental class yields an isomorphism

$$\langle \cdot, [X] \rangle : \mathrm{H}^N(X;\mathbb{Z}) \to \mathbb{Z}.$$

Definition 2.0.5. In the above set-up, if K is a multiplicative sequence and $E \to X$ a complex vector bundle, then the K-genus of E is defined by

$$\Phi_K(E) := \langle K_n(E), [X] \rangle.$$

The case of primary interest to us will be when X is a closed complex manifold of complex dimension n. In this case, when we talk of the total K-class or series or the K-genus of X, we mean the K-class or series or the K-genus of the holomorphic tangent bundle $\mathrm{T}X \to X$. This gives us the Chern, Pontryagin, Todd, L and $\hat{A}$ genera of a closed complex manifold X. Note that $\Phi_p(X) = \Phi_L(X) = \Phi_{\hat{A}}(X) = 0$ unless n itself is even, i.e. $n=2k$ for some $k \geq 0$ and $\dim X = 4k$.[6] Somewhat surprisingly, these genera are often integers, as can be seen from the following fascinating collection of results.

- Let X be a closed complex manifold.

  Theorem 2.0.6 (Gauss-Chern-Bonnet)**.** We have $\Phi_c(X) = \chi(X)$, where $\chi(X)$ is the topological Euler characteristic of X.

- Let X be a smooth projective variety over $\mathbb{C}$.

  Theorem 2.0.7. We have $\Phi_{\mathrm{Td}}(X) = \chi(X, \mathcal{O}_X)$, where $\chi(X,\mathcal{O}_X)$ is the holomorphic Euler characteristic of X.

- Let X be a smooth manifold of dimension $\dim X = 4k$ for some $k \geq 0$. The cup product gives us a symmetric bilinear form on $\mathrm{H}^{2k}(X;\mathbb{R})$; we define the signature $\mathrm{sign}(X)$ of X to be the signature of this bilinear form.

  Theorem 2.0.8 (Hirzebruch Signature Theorem)**.** We have $\Phi_L(X) = \mathrm{sign}(X)$.

[6] The L-genus and $\hat{A}$-genus are defined even for closed real smooth manifolds of dimension $\dim X = 4k$ without a complex structure, since in this context the definition of the Pontryagin classes $p_j(X)$ still makes sense, and that total L- and $\hat{A}$-classes can be expressed in terms of the $p_j(X)$. We did not build the theory from this perspective because we will only deal with closed complex manifolds in this paper.

- Recall that a closed manifold X is said to be spin manifold if it can be given the structure of an oriented Riemannian manifold whose oriented orthogonal frame bundle $\mathrm{SO}_n(X)$ admits a lift to a principal Spin_n-bundle $\mathrm{Spin}_n(X)$. This happens iff the second Stiefel-Whitney class $w_2(X)$ vanishes. In particular, if X is a complex manifold, then this happens iff $c_1(X)$ is even, since in this case $w_2(X) \equiv c_1(X) \pmod 2$. (See [2, §II.1-2].)

  Theorem 2.0.9. Let X be a spin manifold of dimension $\dim X = 4k$ for some $k \geq 0$. Then the $\hat{A}$-genus $\Phi_{\hat{A}}(X)$ is an integer. Further, if $\dim X \equiv 4 \pmod 8$, then $\Phi_{\hat{A}}(X)$ is even.

- Recall that for a Riemmanian manifold X, the scalar curvature $\kappa : X \to \mathbb{R}$ is defined by averaging all sectional curvatures at a given point, i.e. it is the trace of the Ricci curvature tensor.

  Theorem 2.0.10. If X is a spin manifold of everywhere positive scalar curvature κ, i.e. $\kappa > 0$. Then $\Phi_{\hat{A}}(X) = 0$.

We shall note prove any of these here, but remark that they can all be derived from the Atiyah-Singer Index Theorem. Indeed, these observations were part of the impetus that led to the discovery of the Index Theorem. For proofs of all these results, we refer the reader to [2, Ch. IV] and [1, Thm 20.2.2]. In the rest of the paper, we present examples of genera computations for some closed complex manifolds.

2.1 Compact Riemann Surfaces

Let $X = \Sigma_g$ be a Riemann surface of genus $g \geq 0$. Then the integral cohomology $\mathrm{H}^*(X;\mathbb{Z})$ of X is given by

$$\mathrm{H}^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \\ \mathbb{Z}^{2g}, & \text{if } k = 1, \\ 0, & \text{else.} \end{cases}$$

The total Chern class of the tangent bundle $\mathrm{T}X$ is given by

$$c(\mathrm{T}X) = 1 + (2 - 2g)\eta_X.$$

This fact is usually stated by saying that the canonical class K_X has degree $2g-2$, where $K_X = c_1(\mathrm{T}^\vee X)$. It follows that

$$\Phi_c(X) = 2 - 2g \text{ and } \Phi_{\mathrm{Td}}(X) = 1 - g.$$

Note that this computation verifies Theorems 2.0.6 and 2.0.7: indeed, we know that the topological Euler characteristic $\chi(X) = 2 - 2g$, whereas the holomorphic Euler characteristic satisfies

$$\chi(X, \mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - g.$$

(The equality $h^0(\mathcal{O}_X) = 1$ is clear, and by Serre duality we have

$$h^1(\mathcal{O}_X) = h^0(\Omega_X) = g,$$

either by definition of the genus g or by a standard theorem in the theory of curves.) For dimension reasons, we have

$$\Phi_p(X) = \Phi_L(X) = \Phi_{\hat{A}}(X) = 0.$$

2.2 Projective Spaces

For $n \geq 0$, let $\mathbb{P}^n$ be complex projective space of dimension n. Then the cohomology ring of $\mathbb{P}^n$ with coefficients in $\mathbb{Z}$ is given by

$$\mathrm{H}^*(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}[\zeta]/(\zeta^{n+1}), \text{ where } |\zeta| = 2.$$

Here ζ is the Poincare dual to the fundamental class of a hyperplane $[\mathbb{P}^{n-1}] \in \mathrm{H}_{2n-2}(\mathbb{P}^n;\mathbb{Z})$, and the top cohomology $\mathrm{H}^{2n}(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}\eta_{\mathbb{P}^n}$ is generated by $\eta_{\mathbb{P}^n} = \zeta^n$.

On $\mathbb{P}^n$, we have the line bundles $\mathcal{O}_{\mathbb{P}^n}(r)$ for any $r \in \mathbb{Z}$, where $\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{S}$ is the tautological bundle and we have for $r \in \mathbb{Z}$ that $\mathcal{O}_{\mathbb{P}^n}(r) = \mathcal{S}^{\otimes(-r)}$. Further, for any $r \in \mathbb{Z}$, we have

$$c_1(\mathcal{O}_{\mathbb{P}^n}(r)) = 1 + r\zeta.$$

Let $\mathcal{Q}$ be the tautological quotient bundle on $\mathbb{P}^n$. The tangent bundle $\mathrm{T}\mathbb{P}^n$ of $\mathbb{P}^n$ satisfies

$$\mathrm{T}\mathbb{P}^n \cong \mathcal{S}^\vee \otimes \mathcal{Q}.$$

Twisting the tautological sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{Q} \to 0$$

by $\mathcal{S}^\vee = \mathcal{O}_{\mathbb{P}^n}(1)$ yields

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathrm{T}\mathbb{P}^n \to 0.$$

Therefore, multiplicativity allows us to compute the required total K-classes and genera.

- The total Chern class is given by

  $$c(\mathbb{P}^n) = (1+\zeta)^{n+1} = 1 + (n+1)\zeta + \frac{n(n+1)}{2}\zeta^2 + \cdots + (n+1)\zeta^n.$$

  In particular, the Chern genus is

  $$\Phi_c(\mathbb{P}^n) = n+1.$$

  This verifies Theorem 2.0.6 in this case because we know that $\chi(\mathbb{P}^n) = n+1$.
- The total Todd class is given by

  $$\begin{aligned}\mathrm{Td}(\mathbb{P}^n) &= \left(\frac{\zeta}{1-\mathrm{e}^{-\zeta}}\right)^{n+1} \\ &= 1 + \frac{1}{2}(n+1)\zeta + \frac{1}{24}(3n^2+5n+2)\zeta^2 + \frac{1}{48}n(n+1)^2\zeta^3 + \cdots .\end{aligned}$$

  For small values of n, these are

  $$\begin{aligned}\mathrm{Td}(\mathbb{P}^1) &= 1 + \zeta, \\ \mathrm{Td}(\mathbb{P}^2) &= 1 + \frac{3}{2}\zeta + \zeta^2, \\ \mathrm{Td}(\mathbb{P}^3) &= 1 + 2\zeta + \frac{11}{6}\zeta^2 + \zeta^3.\end{aligned}$$

  Note that the Todd genus satisfies

  $$\Phi_{\mathrm{Td}}(\mathbb{P}^n) = 1$$

  for $n = 1, 2, 3$, above; this is true for arbitrary n, and can be explained by Lemma 1.0.7. This verifies Theorem 2.0.7 in this case because we know independently that $h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is 1 for $i = 0$ and 0 for $i \geq 1$, so that $\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 1$.

- The total Pontryagin class

$$p(\mathbb{P}^n) = (1+\zeta^2)^{n+1} = 1 + (n+1)\zeta^2 + \cdots ,$$

where the last term is given by $\binom{n+1}{n/2}\zeta^n$ if $2 \mid n$ and $\binom{n+1}{(n-1)/2}\zeta^{n-1}$ else. In particular, if $n = 2k$ is even, then the Pontryagin genus is given by

$$\Phi_p(\mathbb{P}^{2k}) = \binom{2k+1}{k}.$$

- The total L-class is given by

$$\begin{aligned} L(\mathbb{P}^n) &= \left(\frac{\zeta}{\tanh \zeta}\right)^{n+1} \\ &= 1 + \left(\frac{n+1}{3}\right)\zeta^2 + \left(\frac{5n^2+3n-2}{90}\right)\zeta^4 \\ &\qquad + \left(\frac{35n^3-42n^2-65n+12}{5670}\right)\zeta^6 + \cdots . \end{aligned}$$

For small values of n, this is given by

$$\begin{aligned} L(\mathbb{P}^1) &= 1, \\ L(\mathbb{P}^2) &= 1+\zeta^2, \\ L(\mathbb{P}^3) &= 1+\frac{4}{3}\zeta^2, \\ L(\mathbb{P}^4) &= 1+\frac{5}{3}\zeta^2+\zeta^4, \\ L(\mathbb{P}^5) &= 1+2\zeta^2+\frac{23}{15}\zeta^4, \\ L(\mathbb{P}^6) &= 1+\frac{7}{3}\zeta^2+\frac{98}{45}\zeta^3+\zeta^4. \end{aligned}$$

Note again that the L-genus is given by

$$\Phi_L(\mathbb{P}^{2k}) = 1$$

for $k = 1, 2, 3$ above; again, this is true for arbitrary k, and is explained by the remark at the end of Example 1.0.8. This illustrates the Hirzebruch Signature Theorem (Theorem 2.0.8), because the signature of the intersection form on the one-dimensional middle cohomology $\mathrm{H}^m(\mathbb{P}^{2m}, \mathbb{Z}) = \mathbb{Z}\zeta^m$ is clearly 1.
- Finally, the total $\hat{A}$-class is given by

$$\begin{aligned} \hat{A}(\mathbb{P}^n) &= \left(\frac{\zeta/2}{\sinh(\zeta/2)}\right)^{n+1} \\ &= 1 - \left(\frac{n+1}{24}\right)\zeta^2 + \left(\frac{5n^2+12n+7}{5760}\right)\zeta^4 \\ &\qquad - \left(\frac{35n^3+147n^2+205n+93}{2903040}\right)\zeta^6 + \cdots . \end{aligned}$$

For small values of n, this is given by

$$\begin{aligned}
\hat{A}(\mathbb{P}^1) &= 1, \\
\hat{A}(\mathbb{P}^2) &= 1 - \frac{1}{8}\zeta^2, \\
\hat{A}(\mathbb{P}^3) &= 1 - \frac{1}{6}\zeta^2, \\
\hat{A}(\mathbb{P}^4) &= 1 - \frac{5}{24}\zeta^2 + \frac{3}{128}\zeta^4, \\
\hat{A}(\mathbb{P}^5) &= 1 - \frac{1}{4}\zeta^2 + \frac{1}{30}\zeta^4, \\
\hat{A}(\mathbb{P}^6) &= 1 - \frac{7}{24}\zeta^2 + \frac{259}{5760}\zeta^4 - \frac{5}{1024}\zeta^6.
\end{aligned}$$

Note some values of the $\hat{A}$-genera:

$$\Phi_{\hat{A}}(\mathbb{P}^2) = -1/8, \quad \Phi_{\hat{A}}(\mathbb{P}^4) = 3/128, \quad \Phi_{\hat{A}}(\mathbb{P}^6) = -5/1024.$$

None of these are integers. It follows from Theorem 2.0.9 that $\mathbb{P}^{2k}$ is not spin for $k = 1, 2, 3$. In fact, $\mathbb{P}^n$ is spin iff $n \equiv 1 \pmod 2$ by the condition $w_2(\mathbb{P}^n) = (n+1)\zeta \pmod 2$.

2.3 Grassmannians

More generally, for $n > k \geq 0$, let $X = \mathbb{G}(k,n)$ be the complex Grassmannian of k-planes in $\mathbb{CP}^n$ (or equivalently, $k+1$-dimensional linear subspaces of $\mathbb{C}^{n+1}$); this has complex dimension $N = (k+1)(n-k)$. Then the cohomology ring

$$\mathrm{H}^\bullet(\mathbb{G}(k,n);\mathbb{Z}) = \bigoplus_a \mathbb{Z}\sigma_a$$

is freely generated by the Schubert classes σ_a, where we index over the set of all sequences $a = (a_0, \ldots, a_{n-k})$ of integers satisfying $n-k \geq a_0 \geq a_1 \geq \cdots \geq a_k \geq 0$. Note that the top cohomology group

$$\mathrm{H}^{2N}(\mathbb{G}(k,n);\mathbb{Z}) = \mathbb{Z}\eta_{\mathbb{G}(k,n)}$$

is generated by the Schubert class

$$\eta_{\mathbb{G}(k,n)} = \sigma_{(n-k)^k}.^7$$

On $\mathbb{G}(k,n)$, we have again the tautological bundle $\mathcal{S}$ of rank $k+1$ and the tautological quotient bundle $\mathcal{Q}$ of rank $n-k$, and again the tangent bundle $\mathrm{T}\mathbb{G}(k,n)$ satisfies

$$\mathrm{T}\mathbb{G}(k,n) \cong \mathcal{S}^\vee \otimes \mathcal{Q}. \tag{3}$$

The Chern classes of $\mathcal{S}^\vee$ and $\mathcal{Q}$ are given by

$$c(\mathcal{S}^\vee) = 1 + \sigma_1 + \cdots + \cdots + \sigma_{1^{k+1}} \text{ and } c(\mathcal{Q}) = 1 + \sigma_1 + \cdots + \sigma_{n-k};$$

these can be computed in various ways, but one is to use the explicit description of Chern classes as the cohomology classes corresponding to degeneracy loci of sections (see [3, §5.6.2]). If $\alpha_1, \ldots, \alpha_{k+1}$ are the Chern roots of $\mathcal{S}$ and $\beta_1, \ldots, \beta_{n-k}$ the Chern roots of $\mathcal{Q}$, then the Chern roots of $\mathrm{T}\mathbb{G}(k,n)$ are the $\alpha_i + \beta_j$, and this gives us, in principle, a way to compute the total K-classes and K-genera for $\mathbb{G}(k,n)$. However, the formulae seem to be rather complicated at this level of generality.[8] With the help of Macaulay 2, let's work out some examples in full detail: namely $\mathbb{G}(1,n)$ for $n = 3,4,5$.

[7] Here, as always, $(n-k)^k$ means $n-k, \ldots, n-k$ repeated k times.

[8] You can nonetheless extract some information. For instance, it follows immediately that $c_1(\mathbb{G}(k,n)) = n+1$, so that $\mathbb{G}(k,n)$ is spin iff n is odd.

2.3.1 $\mathbb{G}(1,3)$

Consider $\mathbb{G}(1,3)$, which has dimension $N=4$.

- The total Chern class of $\mathbb{G}(1,3)$ is given by
$$c(\mathbb{G}(1,3)) = 1 + 4\sigma_1 + 7(\sigma_{1,1}+\sigma_2) + 12\sigma_{2,1} + 6\sigma_{2,2},$$
so that in particular the Chern genus is
$$\Phi_c(\mathbb{G}(1,3)) = 6.$$
This verifies Theorem 2.0.6 in this case because
$$\chi(\mathbb{G}(1,3)) = \operatorname{rank}_{\mathbb{Z}} \mathrm{H}^*(\mathbb{G}(1,3);\mathbb{Z}) = 6.$$
- The total Todd class of $\mathbb{G}(1,3)$ is given by
$$\mathrm{Td}(\mathbb{G}(1,3)) = 1 + 2\sigma_1 + \frac{23}{12}(\sigma_{1,1}+\sigma_2) + \frac{7}{3}\sigma_{2,1} + \sigma_{2,2},$$
so that the Todd genus
$$\Phi_{\mathrm{Td}}(\mathbb{G}(1,3)) = 1.$$
This verifies Theorem 2.0.7 in this case because it can be shown independently that
$$\chi(\mathbb{G}(1,3), \boldsymbol{\mathcal{O}}_{\mathbb{G}(1,3)}) = 1;$$
see Remark 1.
- The total Pontryagin class of $\mathbb{G}(1,3)$ is given by
$$p(\mathbb{G}(1,3)) = 1 + 2(\sigma_{1,1}+\sigma_2) + 14\sigma_{2,2},$$
so that the Pontryagin genus is
$$\Phi_p(\mathbb{G}(1,3)) = 14.$$
- The total L-class of $\mathbb{G}(1,3)$ is given by
$$L(\mathbb{G}(1,3)) = 1 + \frac{2}{3}(\sigma_{1,1}+\sigma_2) + 2\sigma_{2,2},$$
so that the L-genus is
$$\Phi_L(\mathbb{G}(1,3)) = 2.$$
This verifies Theorem 2.0.8 in this case, since in the middle cohomology group $\mathrm{H}^4(\mathbb{G}(1,3)) = \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2$, the classes $\sigma_{1,1}$ and σ_2 are orthonormal.
- The total $\hat{A}$-class of $\mathbb{G}(1,3)$ is given by
$$\hat{A}(\mathbb{G}(1,3)) = 1 - \frac{1}{12}(\sigma_{1,1}+\sigma_2),$$
so that that $\hat{A}$-genus
$$\Phi_{\hat{A}}(\mathbb{G}(1,3)) = 0.$$
This verifies Theorem 2.0.9 in this case, since $\mathbb{G}(1,3)$ is spin, as observed above; see also Remark 1.

2.3.2 $\mathbb{G}(1,4)$

Consider $\mathbb{G}(1,4)$, which has dimension $N = 6$.

- The total Chern class of $\mathbb{G}(1,4)$ is given by

$$c(\mathbb{G}(1,4)) = 1 + 5\sigma_1 + (12\sigma_{1,1} + 11\sigma_2) + (30\sigma_{2,1} + 15\sigma_3) \\ + (25\sigma_{2,2} + 35\sigma_{3,1}) + 30\sigma_{3,2} + 10\sigma_{3,3},$$

  so that the Chern genus is

$$\Phi_c(\mathbb{G}(1,4)) = 10.$$

  This verifies Theorem 2.0.6 in this case because

$$\chi(\mathbb{G}(1,4)) = \operatorname{rank}_{\mathbb{Z}} \mathrm{H}^*(\mathbb{G}(1,4);\mathbb{Z}) = 10.$$

- The total Todd class of $\mathbb{G}(1,4)$ is given by

$$\mathrm{Td}(\mathbb{G}(1,4)) = 1 + \frac{1}{2}5\sigma_1 + \frac{1}{12}(37\sigma_{1,1} + 36\sigma_2) + \frac{1}{24}(115\sigma_{2,1} + 55\sigma_3) \\ + \frac{1}{72}(197\sigma_{2,2} + 287\sigma_{3,1}) + \frac{35}{12}\sigma_{3,2} + \sigma_{3,3}.$$

  so that the Todd genus

$$\Phi_{\mathrm{Td}}(\mathbb{G}(1,4)) = 1.$$

  As above, this verifies Theorem 2.0.7 in this case; see Remark 1.
- The total Pontryagin class of $\mathbb{G}(1,4)$ is given by

$$p(\mathbb{G}(1,4)) = 1 + (\sigma_{1,1} + 3\sigma_2) + (15\sigma_{2,2} + 5\sigma_{3,1}) + 35\sigma_{3,3},$$

  so that the Pontryagin genus is

$$\Phi_p(\mathbb{G}(1,4)) = 35.$$

- The total L-class of $\mathbb{G}(1,4)$ is given by

$$L(\mathbb{G}(1,4)) = 1 + \frac{1}{3}(\sigma_{1,1} + 3\sigma_2) + \frac{1}{9}(19\sigma_{2,2} + 4\sigma_{3,1}) + 2\sigma_{3,3},$$

  so that the L-genus is

$$\Phi_L(\mathbb{G}(1,4)) = 2.$$

  This verifies Theorem 2.0.8 in this case, since in the middle cohomology group $\mathrm{H}^6(\mathbb{G}(1,4)) = \mathbb{Z}\sigma_{2,1} \oplus \mathbb{Z}\sigma_3$, the classes $\sigma_{2,1}$ and σ_3 are orthonormal.
- The total $\hat{A}$-class of $\mathbb{G}(1,4)$ is given by

$$\hat{A}(\mathbb{G}(1,4)) = 1 - \frac{1}{24}(\sigma_{1,1} + 3\sigma_2) + \frac{1}{1152}(2\sigma_{2,2} + 17\sigma_{3,1}) - \frac{1}{1024}\sigma_{3,3}.$$

  so that that $\hat{A}$-genus

$$\Phi_{\hat{A}}(\mathbb{G}(1,4)) = -1/1024.$$

  Theorem 2.0.9 tells us that $\mathbb{G}(1,4)$ is not spin, which we knew already from the parity of $c_1(\mathbb{G}(1,4))$.

2.3.3 $\mathbb{G}(1,5)$

Consider $\mathbb{G}(1,5)$, which has dimension $N = 8$.

- The total Chern class of $\mathbb{G}(1,5)$ is given by

$$\begin{aligned} c(\mathbb{G}(1,5)) = 1 &+ 6\sigma_1 + (18\sigma_{1,1} + 16\sigma_2) + (58\sigma_{2,1} + 26\sigma_3) \\ &+ (67\sigma_{2,2} + 91\sigma_{3,1} + 31\sigma_{4,0}) + (120\sigma_{3,2} + 90\sigma_{4,1}) \\ &+ (65\sigma_{3,3} + 105\sigma_{4,2}) + 60\sigma_{4,3} + 15\sigma_{4,4} \end{aligned}$$

  so that the Chern genus is

$$\Phi_c(\mathbb{G}(1,5)) = 15.$$

  This verifies Theorem 2.0.6 in this case because

$$\chi(\mathbb{G}(1,5)) = \operatorname{rank}_{\mathbb{Z}} \mathrm{H}^*(\mathbb{G}(1,5); \mathbb{Z}) = 15.$$

- The total Todd class of $\mathbb{G}(1,5)$ is given by

$$\begin{aligned} \mathrm{Td}(\mathbb{G}(1,5)) = 1 + 3\sigma_1 + \frac{1}{6}(27\sigma_{1,1} + 26\sigma_2) + \frac{1}{2}(17\sigma_{2,1} + 8\sigma_3) \\ + \frac{1}{720}(4325\sigma_{2,2} + 6221\sigma_{3,1} + 1901\sigma_{4,0}) + \frac{1}{120}(953\sigma_{3,2} + 713\sigma_{4,1}) \\ + \frac{1}{720}(2501\sigma_{3,3} + 4301\sigma_{4,2}) + \frac{1}{30}(101\sigma_{4,3}) + \sigma_{4,4} \end{aligned}$$

  so that the Todd genus

$$\Phi_{\mathrm{Td}}(\mathbb{G}(1,5)) = 1.$$

  As above, this verifies Theorem 2.0.7 in this case; see Remark 1.

- The total Pontryagin class of $\mathbb{G}(1,5)$ is given by

$$p(\mathbb{G}(1,5)) = 1 + 4\sigma_2 + (18\sigma_{2,2} + 6\sigma_{3,1} + 6\sigma_4) + (26\sigma_{3,3} + 42\sigma_{4,2}) + 141\sigma_{4,4},$$

  so that the Pontryagin genus is

$$\Phi_p(\mathbb{G}(1,5)) = 141.$$

- The total L-class of $\mathbb{G}(1,5)$ is given by

$$L(\mathbb{G}(1,5)) = 1 + \frac{4}{3}\sigma_2 + \frac{1}{45}(105\sigma_{2,2} + 26\sigma_{3,1} + 26\sigma_4) + \frac{1}{45}(68(\sigma_{3,3} + \sigma_{4,2}) + 3\sigma_{4,4},$$

  so that the L-genus

$$\Phi_L(\mathbb{G}(1,5)) = 3.$$

  This verifies Theorem 2.0.8 in this case, since in the middle cohomology group $\mathrm{H}^8(\mathbb{G}(1,4)) = \mathbb{Z}\sigma_{2,2} \oplus \mathbb{Z}\sigma_{3,1} \oplus \mathbb{Z}\sigma_4$, the classes $\sigma_{2,2}$, $\sigma_{3,1}$, and σ_4 are orthonormal.

- The total $\hat{A}$-class of $\mathbb{G}(1,5)$ is given by

$$\hat{A}(\mathbb{G}(1,5)) = 1 - \frac{1}{6}\sigma_2 + \frac{1}{720}(5\sigma_{2,2} + 11\sigma_{3,1} + 11\sigma_{4,0}) - \frac{1}{720}(\sigma_{3,3} + \sigma_{4,2})$$

  so that the $\hat{A}$-genus

$$\Phi_{\hat{A}}(\mathbb{G}(1,5)) = 0.$$

  This verifies Theorem 2.0.9 in this case since $\mathbb{G}(1,5)$ is spin, as observed above; see also Remark 1.

Remark 1. Here are a few observations from these computations:

(a) We found above that the Todd genera of Grassmannians satisfy

$$\Phi_{\mathrm{Td}}(\mathbb{G}(k,n)) = 1$$

for $k = 1, n = 3, 4, 5$.[9] This is true for general k and n and can be seen from Theorem 2.0.7, the Borel-Weil-Bott Theorem, and the fact that $\mathbb{G}(k,n)$ are complete homogenous spaces: the cohomology groups $\mathrm{H}^i(X, \mathcal{O}_X)$ for any complete homogenous space $X = G/P$ (where G is a semisimple complex Lie group) are all zero for $i > 0$, whereas $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{C}$, so that $\chi(X, \mathcal{O}_X) = 1$ as needed.

(b) We found that the L-genera of the Grassmannians of lines satisfy

$$\Phi_L(\mathbb{G}(1,n)) = \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

for $n = 3, 4, 5$. This is true for general n by Theorem 2.0.8. Indeed, the middle cohomology

$$\mathrm{H}^{2n-2}(\mathbb{G}(1,n)) = \bigoplus_{k=0}^{\lfloor (n-1)/2 \rfloor} \mathbb{Z}\sigma_{n-1-k,k},$$

and it follows from Pieri's formula (see [3, Prop. 4.11]) that these Schubert classes are orthonormal. Similarly, it should be possible to obtain a combinatorial formula for $\Phi_L(\mathbb{G}(k,n))$ in general.

(c) We found above that

$$\Phi_{\hat{A}}(\mathbb{G}(1, 2n+1)) = 0$$

for $n = 1, 2$, even though Theorem 2.0.9 only requires this quantity to be an integer. Again, this is true for general n and can be explained by Theorem 2.0.10 since these Grassmannians have positive scalar curvature under the pullback (or equivalently restriction) of the Fubini-Study metric on $\mathbb{P}^{2n^2+3n}$ by the Plücker embedding, as can be checked explicitly.

[9] The previous section shows that the result is also true for arbitrary n when $k = 0$.

2.4 Hypersurfaces

For integers $d, n \geq 1$, let $X = X_d^n \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Then from the normal sequence

$$0 \to \mathrm{T}X \to \mathrm{T}\mathbb{P}^{n+1}|_X \to \mathcal{N}_{X/\mathbb{P}^{n+1}} \to 0$$

along with the identification

$$\mathcal{N}_{X/\mathbb{P}^{n+1}} \cong \mathcal{O}_X(d),$$

follows that the Chern class of X is given by

$$c(X) = \frac{(1+\zeta_X)^{n+2}}{1+d\zeta_X} = 1 + (n+2-d)\zeta_X + \cdots,$$

where $\zeta_X = \zeta|_X \in \mathrm{H}^2(X;\mathbb{Z})$ is the restriction of the hyperplane class to X. Since ζ_X is the restriction of the standard Kähler class, it follows that it is indivisible by two, and hence that $X = X_d^n$ is spin iff $n \equiv d \pmod 2$. Finally, we have that

$$\zeta_X^n = d\eta_X,$$

since d is the degree of X. Next, from this computation and the Lefschetz Hyperplane Theorem, we can compute the cohomology groups of X as

$$\mathrm{H}^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, 2, \ldots, 2n, \text{except } k = n \text{ if } 2 \mid n, \\ \mathbb{Z}^{b_{n,d}}, & k = n, \\ 0, & \text{else}, \end{cases}$$

where $b_{n,d}$ is the unique integer that makes the topological Euler characteristic

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \binom{n+2}{n-i} d^{i+1}.$$

See [3, §5.7.3] for more details. Let's analyze the case $n = 2$ in more detail. We have

$$b_{2,d} = d^3 - 4d^2 + 6d - 2.$$

We now compute away:

- The total Chern class is given by

$$c(X_d^2) = 1 + (4-d)\zeta_X + (d^3 - 4d^2 + 6d)\eta_X,$$

  so that the Chern genus is

$$\Phi_c(X_d^2) = d^3 - 4d^2 + 6d,$$

  which agrees with $\chi(X_d^2)$, verifying Theorem 2.0.6 in this case.
- The total Todd class of X_d^2 is given by

$$\mathrm{Td}(X_d^2) = 1 + \frac{1}{2}(4-d)\zeta_X + \frac{1}{6}d(d^2 - 6d + 11)\eta_X,$$

so that the Todd genus is

$$\Phi_{\mathrm{Td}}(X_d^2) = \frac{1}{6}d(d^2 - 6d + 11) = 1 + \binom{d-1}{3}.$$

On the other hand, we can also compute $\chi(X, \mathcal{O}_X)$ directly: from the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

of sheaves on $\mathbb{P}^3$, we conclude that

$$\chi(X, \mathcal{O}_X) = \chi(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) - \chi(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) = 1 - \left(-\binom{d-1}{3}\right) = 1 + \binom{d-1}{3},$$

where we have used our knowledge of the cohomology of line bundles on $\mathbb{P}^n$. This verifies Theorem 2.0.7 in this case.

- The total Pontryagin class of X_d^2 is given by

$$p(X_d^2) = 1 - (d^3 - 4d)\eta_X,$$

so that the Pontryagin genus is

$$\Phi_p(X_d^2) = d^3 - 4d.$$

- The total L class of X_d^2 is given by

$$L(X_d^2) = 1 - \frac{1}{3}(d^3 - 4d)\eta_X,$$

so that the L-genus is

$$\Phi_L(X_d^2) = -\frac{1}{3}(d^3 - 4d) = -2\binom{d+1}{3} + d.$$

- The total $\hat{A}$ class of X_d^2 is given by

$$\hat{A}(X_d^2) = 1 + \frac{1}{24}(d^3 - 4d)\eta_X,$$

so that the $\hat{A}$-genus is

$$\Phi_{\hat{A}}(X_d^2) = \frac{1}{24}(d^3 - 4d).$$

This is an integer iff d is even, in which case, we can write

$$\Phi_{\hat{A}}(X_d^2) = 2\binom{(d+2)/2}{3},$$

so that in fact, $\Phi_{\hat{A}}(X_d^2)$ is an even integer. This verifies Theorem 2.0.9 in this case, since X_d^2 is spin iff d is even.

Remark 2. These calculations have very interesting consequences. For instance, while $X_2^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ clearly admits a metric of everywhere positive scalar curvature, the manifolds X_{2k}^2 for $k \geq 2$ do not, by the above calculation and Theorem 2.0.10. Taking $k = 2$ gives us a quartic surface X_4^2, the famous K3 surface, for which we have $b_{2,4} = 22$ and $\Phi_L(X_4^2) = -16$, which along with the fact that the intersection form on $\mathrm{H}^2(X_4^2; \mathbb{Z})$ is even completely determines this middle cohomology group as the lattice $\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^2$. (See [4, Ch. 1., Prop. 3.5]).

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